JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 68, 228-264 (1979)

The Extension and Completion of the Universal Measure and the Dual of the Space of Measures

WILLIAM H. GRAVES

Mathematics Department, Phillips Hall, University of North Carolina, Chapel Hill, North Carolina 27514

AND

DENNIS SENTILLES*

Mathematics Department, University of Missouri, Columbia, Missouri 65201

Submitted by G.-C. Rota

Every vector measure μ factors uniquely by $\mu = \tilde{\mu} \circ \chi$ where χ is a universal measure and $\tilde{\mu}$ is a linear map continuous on simple functions in the strict topology. The completion of χ describes the dual of the space of measures by uniform closure based on χ-measurable sets and also as the completion of the simple functions in the strict topology.

Descriptions of the variation norm dual of the space of bounded measures on a σ-algebra $\mathscr{A}$, as a class of (continuous) functions have been known, and have reappeared for some twenty years now. This paper offers an added explicitness and a simplified, almost classical, interpretation or reconstruction of these descriptions. Its centerpiece is a natural extension of the universal measure χ of previous work combined with a formula for the point value of such functions, the latter implicit in earlier work.

More exactly, if ϕ is a bounded linear functional of measures μ, then ϕ admits a representation by a function $\hat{\phi}$ of points x of a space Y given by

$$\hat{\phi}(x) = \lim_{a \to x} \frac{\phi(\mu_a)}{\mu(a)}$$

with inversion

$$\phi = \int_Y \hat{\phi} \, d\chi$$

* On leave at: Mathematics Department, Phillips Hall, University of North Carolina, Chapel Hill, N.C. 27514.

0022-247X/79/030228-37$02.00/0

and spectral form

$$\phi = \int_{\mathscr{R}} \lambda \, dP_\phi$$

with P_ϕ idempotent valued of the form $P_\phi = \chi\hat{\phi}^{-1}$, the χ distribution of $\hat{\phi}$.

These results, and the paper, originate in the realization that the aforementioned dual is but the completion of the simple functions over $\mathscr{A}$ in the locally convex topologies τ and β of previous work.

The paper is organized as follows. Section 1 consists of preliminaries and a listing of previous results needed herein. Section 2 equates the completion and the bidual of the simple functions on a σ-algebra, gives two naive descriptions of these and some essential examples. Section 3 extends the universal measure to a large complete algebra of "measurable sets". Section 4 describes the bidual and the measurable sets in a special case. Section 5 reduces the general case to that of Section 4, introduces the space Y, obtains its properties and a topological classification of the measurable sets. Section 6 studies the inversion of the functional/function correspondence established in Section 5 and contains the integral and spectral representation theorems. Section 7 characterizes the algebra of measurable sets induced by sets of χ measure zero in various ways.

Preliminaries

A σ-algebra of sets is a σ-complete Boolean algebra under $\cap$ and symmetric difference Δ. We find it no restriction and indeed necessary (see Ex. 2.3) to use the latter interpretation and henceforth $\mathscr{A}$ denotes any σ-complete Boolean algebra with identity e, and S denotes the unique compact Hausdorff totally disconnected topological space that is the Stone space of $\mathscr{A}$. For $a \in \mathscr{A}$ we shall also let a denote the clopen subset of S corresponding to a in the Stone representation and let χ_a denote the associated continuous function on S defined by a. The linear span of $\{\chi_a: a \in \mathscr{A}\}$ in $C(S)$ (the continuous functions on S) is denoted by $\mathscr{S}(\mathscr{A})$ and we call each of its elements a simple function. The mapping χ of $\mathscr{A}$ into the linear space $\mathscr{S}(\mathscr{A})$ defined by $\chi(a) = \chi_a$ is the a finitely additive vector valued measure on $\mathscr{A}$.

The uniform norm topology on $C(S)$ and on $\mathscr{S}(\mathscr{A})$ is defined by the norm $\|f\| = \sup\{|f(s)|: s \in S\}$. By the Stone-Weierstrass and Stone representation theorems, $\mathscr{S}(\mathscr{A})$ is norm dense in $C(S)$.

The topology of emphasis on $\mathscr{S}(\mathscr{A})$ is however not the norm topology but rather the topologies τ and β of previous work [14; 26], for the reason that: (1) these yield as dual to $\mathscr{S}(\mathscr{A})$ the bounded countably additive measures on $\mathscr{A}$, and (2) the completion of $\mathscr{S}(\mathscr{A})$ in these is exactly the variation norm dual of these measures (2.1), the description of which is our aim.

The topology β is defined on $C(S)$, which contains $\mathscr{S}(\mathscr{A})$, in [26] so as to force the countable additivity of measures on $\mathscr{A}$ which represent β-continuous functionals on $\mathscr{S}(\mathscr{A})$. Let $\{a_n\}$ be any increasing sequence ($a_na_{n+1} = a_n$ for all n) in $\mathscr{A}$ with $a = \vee a_n$ ($\vee$ ($\wedge$) denotes supremum (infimum) in the algebra $\mathscr{A}$); we will write $a_n \nearrow a$. In the Stone space S the residue $Q = a\backslash\bigcup_{n=1}^{\infty} a_n$ is closed and nowhere dense and we define a locally convex topology β_Q on $\mathscr{S}(\mathscr{A})$ by seminorms $p_\phi(f) = \|\phi f\|$ where $\phi \in C(S)$ and $\phi \equiv 0$ on Q. The topology β on $\mathscr{S}(\mathscr{A})$ is then defined as the inductive limit of the topologies β_Q taken over all such Q. Each β_Q being a "strict topology", the literature on these is immediately available, (e.g. [2], [4], [6], [7], [24], [27] among others), redeeming somewhat the definition of β.

A principal consequence is that the mapping $\chi\colon \mathscr{A} \to \mathscr{S}(\mathscr{A})_\beta$ is β-countably additive [26, 4.1] and hence the β-dual of $\mathscr{S}(\mathscr{A})$ is identical with the (countably additive) measures μ on $\mathscr{A}$ thru the correspondence $\mu = \hat{\mu} \circ \chi$ where $\hat{\mu}$ is the β-continuous functional $\hat{\mu}(\sum_{i=1}^n \alpha_i\chi(a_i)) = \sum_{i=1}^n \alpha_i\mu(a_i)$, [26, 4.3].

Graves [14] on the other hand, influenced by Rota [21], takes the equation $\mu = \hat{\mu} \circ \chi$ as the point of departure and concentrates on rings of sets $\mathscr{R}$ rather than Boolean algebras $\mathscr{A}$; this is no loss of generality [14, Section 12] but with emphasis here on σ-completeness we adopt the technique of [14] directly to Boolean algebras. If $\mu\colon \mathscr{A} \to W$ where W is a locally convex space and $\mu(a + b) = \mu(a) + \mu(b)$ for all $ab = 0$ in $\mathscr{A}$, and if $a_i \nearrow a$ implies $\mu(a_i) \to \mu(a)$ in W we call μ a vector measure on $\mathscr{A}$ into W. Given such a μ define the linear map $\tilde{\mu}\colon \mathscr{S}(\mathscr{A}) \to W$ by $\tilde{\mu}(\sum_{i=1}^n \alpha_i\chi(a_i)) = \sum_{i=1}^n \alpha_i\mu(a_i)$. Then $\mu = \tilde{\mu} \circ \chi$ and $\tilde{\mu}$ is unique. Define a topology τ_0 on $\mathscr{S}(\mathscr{A})$ to be the weakest locally convex topology on $\mathscr{S}(\mathscr{A})$ making all such maps $\tilde{\mu}$ continuous (linear) maps into W, for all μ and W. The redeeming feature is that $\chi\colon \mathscr{A} \to \mathscr{S}(\mathscr{A})_{\tau_0}$ is a vector measure which is *a priori* the universal vector measure on $\mathscr{A}$ into the category LCS of locally convex spaces and continuous linear maps [14], making vector measure theory virtually a corollary of real measure theory, for as with β, the τ_0 dual of $\mathscr{S}(\mathscr{A})$ is the space of real c.a. measures on $\mathscr{A}$ [14]. For this reason we establish that $\tau_0 = \beta$ on $\mathscr{S}(\mathscr{A})$.

PROPOSITION. *Suppose that $\mathscr{A}$ is a σ-complete Boolean algebra and that for each $a \in \mathscr{A}$, $a \neq 0$, there is a real measure μ on $\mathscr{A}$ such that $\mu(a) \neq 0$. Then $\beta = \tau_0$ on $\mathscr{S}(\mathscr{A})$.*

Proof. From [14; 26] both τ_0 and β have the same dual and since $\chi\colon \mathscr{A} \to \mathscr{S}(\mathscr{A})_\beta$ is a vector measure, the identity map $\mathscr{S}(\mathscr{A})_{\tau_0} \to \mathscr{S}(\mathscr{A})_\beta$ must be continuous by the definition of τ_0. Hence $\beta \leqslant \tau_0$. But for σ-complete $\mathscr{A}$, β is the Mackey topology [26; 4.9] so $\beta = \tau_0$.[1]

[1] We take this opportunity to note that [26, 4.9] is false unless $\mathscr{A}$ is σ-complete. An easy example is the algebra $\mathscr{A}$ of finite/cofinite subsets of $N = \{1, 2,...\}$ and weak * compact set $H = \{0\} \cup \{\delta_{n+1} - \delta_n : n = 1, 2,...\}$; [26; 5.3] remains valid with altered proof. A direct proof that β is Mackey for σ-complete $\mathscr{A}$ can also be given as in [14; 11.7].

With this proposition the universal factorization $\mu = \tilde{\mu} \circ \chi$ and corollary vector-measure theory of [14] can be used in conjunction with the β-theory of [26]. We will use the notation $\mathscr{S}(\mathscr{A})_\beta$ because of the existing β literature; a later paper will study the $\beta - \tau$ connection in the absence of σ-completeness and in more detail.

The results of [14; 26] we believe establish the appropriateness of $\beta - \tau$ to the study of the duality of function and measure despite the unwieldly initial definitions. Indeed on norm bounded sets, we reduce these to the more familiar topology of convergence in measure, and [14] for example achieves the Lebesgue, Hahn and Yosida–Hewitt decompositions easily and in a unified way. With a bit more notation we list further results essential to this paper.

Following [26] we denote the β-dual $\mathscr{S}(\mathscr{A})'_\beta$ (and $C(S)'_\beta$) by $L^1(\mathscr{A})$. The strong topology on $L^1(\mathscr{A})$ is given by the norm $\| \hat{\mu} \|_1 = \sup\{| \hat{\mu}(f)| : f \in \mathscr{S}(\mathscr{A}), \| f \| \leqslant 1\}$ —because the β and $\| \;\|$—bounded sets in $\mathscr{S}(\mathscr{A})$ coincide [14, Section 3; 26, Section 6]. The aim of this paper is to describe the dual $(L^1(\mathscr{A}), \| \;\|_1)'$ and the completion of $\mathscr{S}(\mathscr{A})_\beta$ as closely as one can in terms of $\mathscr{A}$ and its Stone space S; note also that $\| \hat{\mu} \|_1$ is also the total variation of the measure $\mu = \hat{\mu} \circ \chi$ [11].

In $L^1(\mathscr{A})$, set $\hat{\mu} \geqslant \hat{\nu}$ iff $\hat{\mu}(f) \geqslant \hat{\nu}(f)$ for all $f \geqslant 0$ in $\mathscr{S}(\mathscr{A})$; equivalently, $\mu(a) \geqslant \nu(a)$ for all $a \in \mathscr{A}$. Let $L^1(\mathscr{A})^+ = \{\hat{\mu} \geqslant 0\}$. Define also $\hat{\mu}^+(f) = \sup\{\hat{\mu}(g) : 0 \leqslant g \leqslant f\}$ for $f \geqslant 0$ and extend to all f as usual. Write $\hat{\mu}^- = \hat{\mu}^+ - \hat{\mu}$, $| \hat{\mu} | = \hat{\mu}^+ + \hat{\mu}^-$ and denote by $\mu^\pm$ and $| u |$ the corresponding measures; note that $\mu^+(a) = \sup\{\mu(b) : b \leqslant a\}$. For $a \in \mathscr{A}$ define $\hat{\mu}_a(f) = \hat{\mu}(\chi(a) f)$ and more generally for $f \in \mathscr{S}(\mathscr{A})$, define $\hat{\mu}_f(g) = \hat{\mu}(fg)$.

For the sake of clarity the following results from [14; 26] would have to be quoted in the sequel; for conciseness and economy we list them here. From this point on, $\mathscr{A}$ is assumed to meet the hypothesis of 1.1.

THEOREM 1.1 [14; 26]. *Let $\mathscr{A}$ be a σ-complete Boolean algebra and suppose that for each $a \neq 0$ in $\mathscr{A}$ there is a measure μ on $\mathscr{A}$ such that $\mu(a) \neq 0$.*

(a) *Any countably additive vector measure $\mu : \mathscr{A} \to W \in LCS$ factors uniquely by $\mu = \tilde{\mu} \circ \chi$, $\tilde{\mu} : \mathscr{S}(\mathscr{A})_\beta \to W$, linear and continuous. In particular, $\chi : \mathscr{A} \to \mathscr{S}(\mathscr{A})_\beta$ is countably additive.*

(b) *Any such $\mu : \mathscr{A} \to \mathscr{R}$ factors uniquely by $\mu = \hat{\mu} \circ \chi$ with $\hat{\mu} \in L^1(\mathscr{A})$.*

(c) *$(L^1(A), \| \;\|_1)$ is a B-space, an L-space in the order defined above, and a band in $(C(S), \| \;\|)'$.*

(d) *The functionals $\hat{\mu} \in L^1(\mathscr{A})$ correspond uniquely to those countably additive measures $\bar{\mu}$ on the Baire σ-field in S for which $\bar{\mu}(a \backslash \bigcup_{n=1}^\infty a_n) = 0$ for all $a_n \nearrow a$ in $\mathscr{A}$ by the equation $\hat{\mu}(f) = \int_S f \, d\bar{\mu}$.*

(e) *The β-equicontinuity of a set $H \subset L^1(\mathscr{A})$ is equivalent to that of $| H | = \{| \hat{\mu} | : \hat{\mu} \in L^1(\mathscr{A})\}$ or of both $H^\pm = \{\hat{\mu}^\pm : \hat{\mu} \in H\}$.*

(f) *β is Hausdorff and coarser than the $\| \ \|$ topology on $\mathscr{S}(\mathscr{A})$, but these have the same bounded sets.*

(g) *β is the finest locally convex topology agreeing with itself on bounded sets and is given thereon by the seminorms*

$$\| f \|_\mu = | \hat{\mu} | (| f |), \qquad \hat{\mu} \in L^1(\mathscr{A}).$$

(h) *Multiplication is β-separately continuous in $\mathscr{S}(\mathscr{A})$ and jointly continuous on bounded sets. For each $\hat{\mu} \in L^1$, $a \in \mathscr{A}$ and $f \in \mathscr{S}(\mathscr{A})$, both $\hat{\mu}_a$ and $\hat{\mu}_f$ are in L^1.*

(g) *A bounded linear functional $\hat{\mu}$ so $\mathscr{S}(\mathscr{A})$ is in $L^1(\mathscr{A})$ iff $a_n \nearrow a$ in $\mathscr{A}$ implies $\| \hat{\mu}_{a_n} - \hat{\mu}_a \|_1 \to 0$.*

All the above are found in [14] in terms of $\mathscr{S}(\mathscr{A})$ and in [26] in terms of $C(S)$ rather than $\mathscr{S}(\mathscr{A})$, but the $\| \ \|$-density of $\mathscr{S}(\mathscr{A})$ in $C(S)$ handles this detail. Going further,

THEOREM 1.2. *A set $H \subset L^1(\mathscr{A})$ is β-equicontinuous iff H is $\| \ \|_1$ bounded and satisfies either* (a) *or* (b).

(a) [14, 4.1; 26, 4.11]. *For any $a_n \nearrow a$ in $\mathscr{A}$,* $\lim \sup_{n \to \infty, \hat{\nu} \in H} \| \hat{\nu}_{a_n} - \hat{\nu}_a \|_1 = 0$.

(b) [14, 4.1; 26, 7.9, 8.1]. *There is a $\hat{\mu} \in L^1(\mathscr{A})^+$ such that H is uniformly μ-continuous.*

Since $| \mu | (a) = \| \hat{\mu}_a \|_1$, β-equicontinuity is equivalent then to uniform countable additivity in variation. Consequently we have the quite useful

THEOREM 1.3 [14, 11.8; 26, 5.3]. *β is the topology of uniform convergence over weak * compact sets in $L^1(\mathscr{A})^+$.*

The proof is the observation that H β-equicontinuous implies $H^\pm$ is β-equicontinuous by 1.2.

THEOREM 1.4 [14, 7.5, 7.6; 26, Section 7]. *If $\mathscr{I}$ is a σ-ideal in $\mathscr{A}$ and $\pi: \mathscr{A} \to \mathscr{A}/\mathscr{I} = \mathscr{B}$ is the quotient homomorphism, then π induces $R: \mathscr{S}(\mathscr{A})_\beta \to \mathscr{S}(\mathscr{B})_\beta$ by $R_{\chi_{\mathscr{A}}}(a) = \chi_{\mathscr{B}}(\pi a)$. R is onto and continuous with isometric adjoint $R'\nu = \nu \circ \pi$ (where $\nu = \hat{\nu} \circ \chi_{\mathscr{B}}$).*

Most importantly of all,

THEOREM 1.5 (Bartle–Dunford–Schwartz) [14, 11.6; 26, Section 7]. *The weak * compact, weakly compact, weak * countably compact and closed β-equicontinuous sets H in $L^1(\mathscr{A})$ coincide.*

2. $L^1(\mathscr{A})'$ AND THE COMPLETION OF $S(\mathscr{A})_\beta$: PRELIMINARY RESULTS

The main point here is that these two coincide, a result found in both [14] and [26], but of basic importance and we repeat it here. We will detail another naive description of the completion as well, leaving deeper results to Sections 5 and 6. Throughout, $\hat{W}$ denotes the completion, and W' the dual, of any $W \in LCS$.

THEOREM 2.1. $\widehat{\mathscr{S}(\mathscr{A})}_\beta = \widehat{C(S)}_\beta = (L^1(\mathscr{A}), \|\ \|_1)'$.

Proof. Grothendieck's completion theorem identifies $\widehat{\mathscr{S}(\mathscr{A})}_\beta$ as the set of all linear forms ϕ on $L^1(\mathscr{A})$ which are weak * continuous on β-equicontinuous sets. By 1.5 $L^1(\mathscr{A})' \subset \mathscr{S}(\mathscr{A})_\beta$. Conversely, any ϕ w^* continuous on β-equicontinuous sets must be bounded, for if $\{\hat{\mu}_n\}$ is bounded in $L^1(\mathscr{A})$, $\{\hat{\mu}_n/n\}$ is β-equicontinuous by 1.2.

Henceforth we denote by $L^\infty(\mathscr{A})$ the space $L^1(\mathscr{A})' = \widehat{\mathscr{S}(\mathscr{A})}_\beta$ along with the completion topology β of uniform convergence over weak * compacta in $L^1(\mathscr{A})^+$ (1.3). This use of the notation $L^\infty(\mathscr{A})$ is a *departure* from that of [26], where $L^\infty(\mathscr{A})$ denotes $C(S)$. In the light of the results herein the notation in [26] appears premature, and $L^\infty(\mathscr{A}) \equiv \widehat{\mathscr{S}(\mathscr{A})}_\beta$ is decidedly more appropriate as the nearly classical description of $L^\infty(\mathscr{A})$ which follows will show. (However, for complete algebras $\mathscr{A}$, particularly measure algebras, $L^\infty(\mathscr{A})$ is $C(S)$, and this is the notation of [11] as well).

As was perhaps first noted by Kaplan [18], as an ordered space $L^\infty(\mathscr{A})$ is much better behaved than one might expect. For ϕ, $\psi \in L^\infty(\mathscr{A})$ we define $\phi \geqslant \psi$ iff $\phi(\hat{\mu}) \geqslant \psi(\hat{\mu})$ for all $\hat{\mu} \in L^1(\mathscr{A})^+$. While 2.1 makes $L^\infty(\mathscr{A})$ very large, a first step in showing that it is reasonably approximable by $\mathscr{S}(\mathscr{A})$ now follows; bounded in $L^\infty(\mathscr{A})$ refers to boundedness in the dual norm.

THEOREM 2.2. *If $\{\phi_\alpha\}$ is a bounded increasing (or decreasing) net in $L^\infty(\mathscr{A})$, then $\phi = \beta\text{-}\lim_\alpha \phi_\alpha$ exists in $L^\infty(\mathscr{A})$. Consequently, $L^\infty(\mathscr{A})$ is a conditionally complete lattice with $\vee\phi_\alpha = \beta\text{-}\lim \phi_\alpha$.*

Proof. The pointwise limit $\phi(\hat{\mu}) = \lim_\alpha \phi_\alpha(\hat{\mu})$ exists on $L^1(\mathscr{A})^+$ and hence in $L^1(\mathscr{A})$. ϕ is of course linear and bounded by hypothesis, hence in $L^\infty(\mathscr{A})$ by 2.1. By Dini's classical "monotone implies uniform" convergence theorem, $\phi_\alpha \to^\beta \phi$ by 1.5 and 1.3. The rest is clear.

Let $\langle\ ,\ \rangle$ denote the action of the dual pair W, $W' \in LCS$. In [26], β is defined at once on all of $C(S)$, since for a σ-algebra of sets $\mathscr{A}$, $C(S)$ represents all bounded $\mathscr{A}$-measurable functions. Here we emphasize the space $\mathscr{S}(\mathscr{A})$. It is convenient however to regard $C(S)$ as a subspace of $L^\infty(\mathscr{A})$ using 2.1 or 2.2 and the natural formula, $\langle f, \hat{\mu}\rangle = \int_S f\, d\bar{\mu}$ where $f \in C(S)$ and (1.1(d)) $\bar{\mu}$ is the Baire countably additive extension of $\mu = \hat{\mu} \circ \chi$ on the clopen sets of S. As we shall see, μ has a

Borel extension (3.7) as well, as a consequence of 2.2. Note that $\langle f, \hat{\mu}\rangle = \sup\{\hat{\mu}(s): 0 \leqslant s \leqslant f, s \in \mathscr{S}(\mathscr{A})\}$, $f \geqslant 0$ in $C(S)$, $\hat{\mu} \in L^1(\mathscr{A})'^+$.

The next examples show that $L^\infty(\mathscr{A})$ is very large and contains some commonly occuring limits.

EXAMPLE 2.3(a). Let $\mathscr{A}$ denote the Borel σ-field of a Hausdorff topological space X. Let $\{f_\alpha\}$ be a bounded increasing net of $\mathscr{A}$-measurable simple functions and $\bar{\mu}$ a measure on $\mathscr{A}$. Then $\bar{\mu}$ defines a β-continuous functional $\mu(s) = \int_X s\, d\mu$, $s \in \mathscr{S}(\mathscr{A})$, and the pointwise limit

$$\phi(\hat{\mu}) = \lim_\alpha \int_X f_\alpha \, d\bar{\mu}$$

defines an element of $L^\infty(\mathscr{A})$ by 2.2.

(b) Let X_d denote the set X with discrete topology, S the Stone space of $\mathscr{A}$, and $\theta: X_d \to S$ be given by $\theta(x)(a) = \chi_a(x)$; $\theta(x)$ is a homomorphism of $\mathscr{A}$. θ is continuous and hence has a unique extension to the Stone–Cech compactification βX_d onto S. Any $\hat{\mu} \in L^1(\mathscr{A})$ defines a bounded linear functional on $C(S)$ with a unique regular Borel measure representation μ^* on S; $\hat{\mu}(f) = \int_S f\, d\mu^*$. Let $K = \theta(\beta X_d \backslash X_d)$, a compact subset of S. Then $\phi(\hat{\mu}) = \mu^*(K)$ is in $L^\infty(\mathscr{A})$ (2.1). It is a useful exercise to work out further details of this example; e.g. any measure $\bar{\mu}$ on $\mathscr{A}$ such that $\bar{\mu}\{x\} = 0$ for all $x \in X$ necessarily lives only on subsets of $\theta(\beta X_d \backslash X_d)$ as a measure μ^* over S.

(c) Let $f \geqslant 0$ be *any* bounded function on X. Let $\mathscr{F}$ denote the finite sets in X. Then $\phi = \lim_{F \in \mathscr{F}} \sum_{x \in F} f(x)\, \chi(\{x\}) \in L^\infty(\mathscr{A})$ by 2.2, where $\mathscr{F}$ is ordered by iinclusion. Note that $\sup\{f(x): x \in X\} = \| \phi \|$ so that $L^\infty(\mathscr{A})$ contains an isometric copy of $\ell^\infty(X)$, *as well as* functionals such as those in (b).

These examples are very important, particularly (b) and (c), in that they show that even if one begins with a σ-algebra $\mathscr{A}$ of subsets of a set X there is no hope of describing all of $L^1(\mathscr{A})'$ in terms of X alone. We must consider a larger set, and we will see that the Stone space S is large enough.

There is a natural multiplication on $L^\infty(\mathscr{A})$ of equal importance with the natural order. It is a special case of the Arens multiplication on a W^*-algebra, and is defined as follows:

Remark 2.4(a). For $\hat{\mu} \in L^1$ and $f \in \mathscr{S}(\mathscr{A})$, $\hat{\mu}_f \in L^1$ by 1.1(h). If $\phi \in L^\infty(\mathscr{A})$ define $\hat{\mu}_\phi$ by

$$\hat{\mu}_\phi(f) = \phi(\hat{\mu}_f).$$

Then $\hat{\mu}_\phi \in L^1(\mathscr{A})$, for $\mu = \hat{\mu}_\phi \circ \chi$ is a measure on $\mathscr{A}$ since $\mu(a) = \phi(\hat{\mu}_a)$, ϕ is $\| \ \|_1$-continuous, and $a_n \nearrow a$ implies $\| \hat{\mu}_{a_n} - \hat{\mu}_a \|_1 \to 0$ by 2.1(a).

(b) Now for $\phi, \psi \in L^\infty(\mathscr{A})$ define $\phi \cdot \psi$ on $L^1(\mathscr{A})$ by

$$(\phi \cdot \psi)\,\hat{\mu} = \phi(\hat{\mu}_\psi).$$

If $\|\hat{\mu}_n\|_1 \to 0$ then $\|(\hat{\mu}_n)_\psi\|_1 \to 0$ since

$$\begin{aligned}\|(\hat{\mu}_n)_\psi\|_1 &= \sup\{|\psi((\hat{\mu}_n)_f)| : f \in \mathscr{S}(\mathscr{A}), \|f\| \leqslant 1\}\\ &\leqslant \|\psi\|\,\|\hat{\mu}_n\|_1 .\end{aligned}$$

Hence $\phi \cdot \psi \in L^\infty(\mathscr{A})$.

It readily follows that $\phi\psi = \psi\phi$ and that $L^\infty(\mathscr{A})$ is a commutative Banach algebra with identity $1 = \chi(e)$. Hence $L^\infty(\mathscr{A})$ is a space $C(Z)$, Z compact Hausdorff; $C(Z)$ is then also the dual of the Banach lattice $L^1(\mathscr{A})$ and by Dixmier's well-known result [9], Z is extremally disconnected.

Thus $L^\infty(\mathscr{A})$ is a space $C(Z)$ as a B-algebra; one can of course use Kakutani's M-space representation as well. Hence our aim of describing $L^\infty(\mathscr{A})$ as a space of functions is already broadly met. We will not use the above results but conduct a "bootstrap" operation in terms of $\mathscr{A}$, S and χ which we hope the sequel justifies. Ultimately we describe Z by S and give a "pointwise" functional/function representation with (spectral) inversion.

The intriguing and puzzling aspect of describing $L^\infty(\mathscr{A})$ as a space of functions can now be considered. The Boolean angebra $\mathscr{A}$ has a Boolean completion $\hat{A}$ with Stone space $\hat{S}$ which is of course the Gleason projective resolution of S. Immediately one suspects that $L^\infty(\mathscr{A}) \cong C(\hat{S})$ (or $Z \cong \hat{S}$) might hopefully result; it generally does not. The ultimate reason is that the natural imbedding $\chi: \mathscr{A} \to \mathscr{S}(\mathscr{A}) \subset L^\infty(\mathscr{A})$ does not imbed $\mathscr{A}$ as a *regular* subalgebra [28, p. 93] of the complete Boolean algebra (7.2) of idempotents in $L^\infty(\mathscr{A})$. Thus the Gelfand/Kakutani reprresentation space Z above is not generally the Gleason space $\hat{S}$. We consider a second example illustrating these remarks and the need for 3(b), Section 3.

EXAMPLE 2.5(a). Let $\mathscr{A}$ denote the Borel algebra of the unit interval $X = [0, 1]$ and let m denote Lebesgue measure. The increasing net $\{\chi(F)\} \subset \mathscr{S}(\mathscr{A})$, where $F \subset [0, 1]$ is finite, has a limit $\phi_0 \in L^\infty(\mathscr{A})$ (2.2). Now $\{F : F \text{ finite}\}$ has the supremum $e = [0, 1]$ in $\mathscr{A}$, but $\phi_0 \neq \chi(e)$ for $\phi_0(\hat{m}) = \lim_F m(F) = 0$, where $m = \hat{m} \circ \chi$.

(b) Applying 2.3(b) to (a) above let $\phi(\hat{\mu}) = \mu^*(K)$, $K = \theta(\beta[0, 1]_d \backslash [0, 1])$. Then $\phi = 1 - \phi_0$ and $\phi(\hat{m}) = \hat{m}(\chi(e)) = 1$. In fact, the Borel representation m^* of m as a measure on S has all its support in K. This observation is both the source of difficulty and the key to the description of $L^\infty(\mathscr{A})$ (3(b)).

(c) The Boolean completion $\hat{A}$ is $2^{[0,1]}$ with $\hat{S} = \beta[0, 1]_d$. If $g \in C(S)$ and $0 \leqslant g \leqslant \phi$ in the order on $L^\infty(\mathscr{A})$, then $g = 0$, so we cannot approximate ϕ by $\mathscr{S}(\mathscr{A})$ from below, and $\hat{S}$ cannot be the space Z.

(d) Referring to 2.3(c) note that for the natural imbedding of $\ell^\infty[0, 1]$ in $L^\infty(\mathscr{A})$, one has $\phi_Q \cdot \ell^\infty = \{0\}$ in the Arens multiplication, where $\phi_Q(\hat{\mu}) = \mu^*(\theta(Q))$ for any fixed compact set $Q \subset \beta[0, 1]\backslash[0, 1]$.

Now a second naive representation of $L^\infty(\mathscr{A})$ in terms of $L^1(\mathscr{A})$. For $\phi \in L^\infty(\mathscr{A})$ define $T_\phi: L^1(\mathscr{A}) \to L^1(\mathscr{A})$ by $T_\phi \hat{\mu} = \hat{\mu}_\phi$ using 2.4(a). Note that $\| T_\phi \| = \| \phi \|$ since $\| \hat{\mu}_\phi \|_1 \leqslant \| \phi \| \| \hat{\mu} \|_1$ and $| \phi(\hat{\mu})| \leqslant | \hat{\mu}_\phi(1)|$. From 2.4(b), $T_\phi T_\psi = T_\psi T_\phi$ so that $\mathscr{B} = \{T_\phi: \phi \in L^\infty(\mathscr{A})\}$ is a commutative B-algebra of linear operators in $L^1(\mathscr{A})$ isometric with $L^\infty(\mathscr{A})$ under $\phi \to T_\phi$. Among these operators are found the projections $P_a \hat{\mu} = \hat{\mu}_a$ defined by elements of $\mathscr{A}$. In fact

THEOREM 2.6. *$\mathscr{B}$ is the largest algebra of bounded operators in $L^1(\mathscr{A})$ containing and commuting with $\{P_a: a \in \mathscr{A}\}$. In other words $\mathscr{B}$ is the closed linear span of $\{P_a\}$ in the strong operator topology. In particular $\mathscr{B}$ is itself commutative and algebraically isomorphic isometric to $L^\infty(\mathscr{A})$.*

Proof. For the first part we only need show that if $T: L^1(\mathscr{A}) \to L^1(\mathscr{A})$ is a bounded operator such that $TP_a = P_a T$ for all $a \in \mathscr{A}$, then there is a $\phi \in L^\infty(\mathscr{A})$ such that $T = T_\phi$. Now the adjoint $T': L^\infty(\mathscr{A}) \to L^\infty(\mathscr{A})$ by 2.1 and since $1 \in \mathscr{S}(\mathscr{A})$, $\phi = T'1$ exists in $L^\infty(\mathscr{A})$. Now if $f \in \mathscr{S}(\mathscr{A})$, $f = \sum_{i=1}^n \alpha_i \chi(a_i)$, then $\hat{\mu}_f = \sum_1^n \alpha_i P_{a_i} \hat{\mu}$, so that $T\hat{\mu}_f = (T\hat{\mu})_f$, whence $\langle T\hat{\mu}, f \rangle = \langle (T\hat{\mu})_f , 1 \rangle = \langle T\hat{\mu}_f , 1 \rangle = \langle \hat{\mu}_f , T'1 \rangle = \langle \hat{\mu}_f , \phi \rangle = \langle \hat{\mu}_\phi , f \rangle = \langle T_\phi \hat{\mu}, f \rangle$ so that $T = T_\phi$.

For the second part, given $\phi \in L^1(\mathscr{A})' = \widehat{\mathscr{S}(\mathscr{A})}_\beta$, there exists $\{f_\alpha\} \subset \mathscr{S}(\mathscr{A})$ such that $f_\alpha \to \phi$ at least pointwise on $L^1(\mathscr{A})$ which puts T_ϕ in the s.o.t. closure as desired. The commutativity of $\mathscr{B}$ then follows.

Notice that the operator norm closure of $\{\sum_{i=1}^n \alpha_i P_{a_i}: a_i \in \mathscr{A}, \alpha_i \text{ real}\}$ is of course $C(S)$ as a B-algebra and thus generally a norm closed proper subspace of $L^\infty(\mathscr{A})$. This is a misleading indication of how close $L^\infty(\mathscr{A})$ is to being a uniform closure of simple functions, for with but the addition of two particular β-limits founded on 2.2 to the class $\{P_a: a \in \mathscr{A}\} \simeq \chi(\mathscr{A})$, uniform closure will suffice (6.2(a)). These two limits are found by applying the Caratheodory process to the universal measure χ with range in the ordered space $L^\infty(\mathscr{A})$ and to this we now turn.

3. EXTENSION OF χ TO 2^S

In this section we define $\chi(E)$ for all sets $E \subset S$, then restrict χ to a class of "measurable sets"; the associated class of projections by these measurable sets will be the enlarged class of projections just mentioned. In Sections 4, 5 we will characterize the measurable sets topologically and in Section 7 show how the same arise from the abstract L-structure of $L^1(\mathscr{A})$ as well.

If U is an open subset of S, the set $I = \{a \in \mathscr{A}: a \subset U\}$ is an ideal in $\mathscr{A}$ and the limit: $\lim_{a \in bI} \chi(a)$ was used for example in [14] to obtain the Lebesgue, Hahn, and Yosida–Hewitt decomposition theorems. We now carry this process further, first defining

$$\chi(U) = \beta - \lim_{a \subset U} \chi(a) \in \widehat{\mathscr{S}(\mathscr{A})}_{\beta} \tag{3(a)}$$

which exists by 2.2 since $\{\chi(a)\}_{a \in I}$ is increasing over a.

Now for any $E \subset S$, $\{\chi(U) \colon U \supset E, U \text{ open}\}$ is a bounded decreasing net in $L^{\infty}(\mathscr{A})$ under set inclusion.

Hence again by 2.2,

$$\chi(E) \equiv \beta - \lim_{U \supset E} \chi(U) \tag{3(b)}$$

exists in $L^{\infty}(\mathscr{A})$. The example ϕ of 2.4(c) illustrates the necessity for this second β-limit, but to reiterate, we shall ultimately see that with this, only norm limits will then be needed to obtain all of $L^{\infty}(\mathscr{A})$ (Section 6).

THEOREM 3.1. *If $\phi = \beta$-$\lim \phi_{\alpha}$ in $L^{\infty}(\mathscr{A})$ and $H \subset L^1(\mathscr{A})$ is β-equicontinuous, then*

$$\lim_{\alpha} \sup_{\hat{\mu} \in H} \| \hat{\mu}_{\phi_{\alpha}} - \hat{\mu}_{\phi} \|_1 = 0.$$

Proof. Recall that $\hat{\mu}_{\phi}$ is defined by $\hat{\mu}_{\phi}(f) = \phi(\hat{\mu}_f)$, so that

$$\| \hat{\mu}_{\phi} \|_1 = \sup\{| \phi(\hat{\mu}_g)| \colon g \in \mathscr{S}(\mathscr{A}), \| g \| \leqslant 1\}.$$

Now,

$$\bigcup_{\hat{\mu} \in H} \{\hat{\mu}_g \colon g \in \mathscr{S}(\mathscr{A}), \| g \| \leqslant 1\}$$

is also β-equicontinuous. For, using 1.2(a), if $a_n \nearrow a$ in $\mathscr{A}$ and $\| g \| \leqslant 1$ then $\|(\hat{\mu}_g)_{a_n} - (\hat{\mu}_g)_a \|_1 \leqslant \| g \| \| \hat{\mu}_a - \hat{\mu}_{a_n} \|_1 \to 0$ uniformly in g. The result now follows from 2.1.

A standard product and uniform convergence argument then shows

COROLLARY 3.2. *Multiplication in $L^{\infty}(\mathscr{A})$ is β-separately continuous in $L^{\infty}(\mathscr{A})$ and jointly continuous on bounded sets.*

COROLLARY 3.3. *If $E \subset F$, then $\chi(F)\,\chi(E) = \chi(E)$.*

Proof. Note that $\chi(E) \leqslant \chi(e) = 1$ for all $E \subset S$ so that (3.2) multiplication is jointly continuous in $\chi(2^S)$. Now $\chi(a)\,\chi(b) = \chi(ab)$ for all $a, b \in \mathscr{A}$. From these two facts follow $\chi(U)\,\chi(V) = \chi(V)$ for $V \subset U$ and both open. Then for arbitrary $E \subset F$, joint continuity implies $\chi(F)\,\chi(E) = \chi(E)$ since $\{V \cap U \colon E \subset V \text{ open}\}$, where $F \subset U$, U open, is cofinal in $\{V' \colon E \subset V' \text{ open}\}$.

An analogue of the next result is [16, p. 46].

For $\hat{\mu} \in L^1(\mathscr{A})$ and $E \subset S$ let $\hat{\mu}_E = \hat{\mu}_{\chi(E)}$ where of course (2.4(a)) $\hat{\mu}_{\chi(E)}(f) = \langle \chi(E), \hat{\mu}_f \rangle$, $f \in \mathscr{S}(\mathscr{A})$.

THEOREM 3.4. (a) *$\hat{\mu}_E \in L^1(\mathscr{A})$ for any $E \subset S$.*

(b) *The formula $P_E\hat{\mu} = \hat{\mu}_E$ is a projection in $L^1(\mathscr{A})$ commuting with $\{P_a: a \in \mathscr{A}\}$ (2.6) and $\chi(E) = P'_E 1$ where P'_E is the adjoint in $L^\infty(\mathscr{A})$.*

Proof. (a) This is 2.4(a)

(b) Referring to 2.6, P_E is just T_ϕ with $\phi = \chi(E)$. Now by 3.3, $\chi(E)\,\chi(E) = \chi(E)$ so $P_E{}^2 = P_E$ by 2.6. The proof of 2.6 shows that $\chi(E) = P'_E 1$.

DEFINITION 3.5. A set $E \subset S$ will be called *measurable* if $\chi(E) + \chi(S \backslash E) = 1$. Let $\mathscr{M}$ denote the class of measurable sets.

Thus $E \in \mathscr{M}$ iff for any $\hat{\mu} \in L^1(\mathscr{A})$, $\hat{\mu}_E + \hat{\mu}_{S \backslash E} = \hat{\mu}$ or equivalently that $I - P_E = P_{S \backslash E}$ where $I = P_e$. In Section 4 and 5 we characterize the measurable sets topologically, and in Section 6, show that $\chi(\mathscr{M})$ generates $L^\infty(\mathscr{A})$ by $\| \,\|$-closure.

THEOREM 3.6. (a) *$\mathscr{M}$ is a σ-algebra of subsets of S containing at least all Borel sets.*

(b) *$\chi: \mathscr{M} \to L^\infty(\mathscr{A})_\beta$ is a countably additive measure.*

(c) *$E \in \mathscr{M}$ iff $\chi(A \cap E) + \chi(A \cap S \backslash E) = \chi(A)$ for all $A \subset S$.*

(d) *$\chi(E)\,\chi(F) = \chi(E \cap F)$ for all $F \in \mathscr{M}$ and $E \subset S$.*

(e) *If $\chi(F) = 0$, then $F \in \mathscr{M}$.*

(f) *$E \in \mathscr{M}$ iff χ is regular at E; that is,*

$$\chi(E) = \vee\{\chi(K): K \subset E \text{ and compact}\} = \beta - \lim_{K \subset E} \chi(K)$$

in the order and topology on $L^\infty(\mathscr{A})$.

It is possible to give an elegant vector-valued proof for 3.6 (see [14]) but many of the details are tedious replicas of familiar real-valued measure arguments. To get on with things we instead exploit the fact that $\chi(E)$ is a functional on measure $\mu(\hat{\mu})$ and that the equation $\chi(E) + \chi(S \backslash E) = 1$ can be verified pointwise on $L^1(\mathscr{A})^+$ and depend on familiar real-valued theory.

Proof. For $\hat{\mu} \in L^1(\mathscr{A})^+$ let $\mu^*(E) = \langle \chi(E), \hat{\mu} \rangle$ for $E \in 2^S$. Since $\mu^*(a) = \mu(a)$ for a clopen in S it is easy to see that μ^* is outer measure on 2^S. Let $\mathscr{M}_\mu$ denote the σ-algebra of μ^* measurable sets. Hence $E \in \mathscr{M}_\mu$ iff $\langle 1, \hat{\mu} \rangle = \mu^*(S) = \mu^*(E) + \mu^*(S \backslash E) = \langle \chi(E), \hat{\mu} \rangle + \langle \chi(S \backslash E), \hat{\mu} \rangle$, so that $\mathscr{M} = \bigcap_{\hat{\mu} \in L^{1+}} \mathscr{M}_\mu$ is a σ-algebra. Since each $\hat{\mu}$ is bounded this also proves (c).

We give a vector argument that open sets and hence Borel sets lie in $\mathscr{M}$. Let N be a β-neighborhood of zero and let U be open in S. Find open $V_0 \supset S \backslash U$ so that $\chi(V) - \chi(S \backslash U) \in N$ for all $V \subset V_0$, $V \supset S \backslash U$. Now find clopen $a \subset U$ and containing the compact set $S \backslash V_0$ so that $\chi(U) - \chi(a) \in N$. Then $V_0 \supset e - a \supset$

$S\backslash U$, $e - a$ is open and $1 = \chi(a) + \chi(e - a)$. This puts $1 - [\chi(U) + \chi(S\backslash U)] \in 2N$, which suffices to prove (a).

Turning to (b), each μ^* is countably additive on $\mathscr{M}_\mu$ whence $E \to \chi(E)$ is countably additive on $\mathscr{M} = \cap \mathscr{M}_\mu$ pointwise on $L^1(\mathscr{A})^+$. But β is uniform convergence over weak compacta in $L^1(\mathscr{A})^+$ (1.3 and 1.5) and monotone pointwise convergence is uniform on compacta since the limit $\chi(\bigcup_{n=1}^\infty E_n)$, being in $L^\infty(\mathscr{A})$, is weakly continuous.

For (d), from 3.3, $\chi(E)\,\chi(F) = \chi(E)$ for $E \subset F$. Hence $\chi(F)\,\chi(E \cap F) = \chi(E \cap F)$ for all E and F. Since $\chi(E) \geqslant \chi(E \cap F)$ this means $\chi(E)\,\chi(F) \geqslant \chi(E \cap F)$. Suppose there is a $\hat{\mu} \in L^1(\mathscr{A})^+$ such that $\chi(E)\,\chi(F)\,\hat{\mu} > \chi(E \cap F)\,\hat{\mu}$. Then also $\chi(E)\,\chi(S\backslash F)\,\hat{\mu} \geqslant \chi(E \cap S\backslash F)\,\hat{\mu}$ whence if F is measurable we obtain

$$\chi(E)\,\hat{\mu} = \chi(E)\,[\chi(F) + \chi(S\backslash F)]\,\hat{\mu} > [\chi(E \cap F) + \chi(E \cap S\backslash F)]\,\hat{\mu}$$

contradicting (c).

(e) Since $\chi(E) + \chi(S\backslash E) \geqslant 1$ for any E this is clear.

(f) $E \in \mathscr{M}$ iff $\chi(E) = 1 - \chi(S\backslash E) = 1 - \lim_{V \supset S/E} \chi(V) = \lim_{V \supset S/F} 1 - \chi(V) = \lim_{V \supset S/E} \chi(S\backslash V) = \lim_{K \subset E} \chi(K)$ since the open set $V \in \mathscr{M}$ by (a), and $S\backslash V$ is compact. By 2.2, $\chi(E) = \vee \chi(K)$, K compact in E as well.

COROLLARY 3.7. *Each $\hat{\mu} \in L^1(\mathscr{A})$ defines a regular Borel measure μ on S by $\bar{\mu}(E) = \langle \chi(E), \hat{\mu} \rangle$. A regular Borel measure $\bar{\mu}$ on S defines a $\hat{\mu} \in L^1(\mathscr{A})$ iff $\bar{\mu}(a \backslash \cup a_n) = 0$ for all $a_n \nearrow a$ in $\mathscr{A}$. Moreover $\bar{\mu}$ is uniquely defined on $\mathscr{M}$ by its values $\mu(a)$, $a \in \mathscr{A}$.*

This follows from 1.1 and 3.6, and the definition of $\chi(E)$. Henceforth $\bar{\mu}$ denotes the regular Borel measure $\bar{\mu} = \hat{\mu} \circ \chi \equiv \langle \chi, \hat{\mu} \rangle$.

Much of the remainder of the paper is aimed at finding a good description of the entire completion $L^\infty(\mathscr{A})$ of $S(\mathscr{A})_\beta$ in terms on the subset $\chi(\mathscr{M})$. (In fact we will see that, as a set $\chi(\mathscr{M}) = \chi(2^S)$.) Before going on we want to unify the various interpretations of $\chi(E)$, $E \subset S$.

Firstly, $\chi(E) \in L^\infty(\mathscr{A}) = \widehat{\mathscr{S}(\mathscr{A})_\beta} = L^1(\mathscr{A})'$, and $\chi(E)$ also defines the projection P_E on $L^1(\mathscr{A})$ of 3.4. But $\chi(E)$ equally well defines a projection $Q_E: L^\infty(\mathscr{A}) \to L^\infty(\mathscr{A})$ by $Q_E\phi = \chi(E)\,\phi$ in the Arens multiplication, as follows from 3.2. Note that $P_E'1 = \chi(E) = Q_E 1$, or more generally, that $P_E' = Q_E$. We find it easier and convenient to dispense with these varied notations and simply write $\chi(E)$ for any one of them, with the context carrying the interpretation. In summary, one can think of χ as a measure on $\mathscr{M}$ whose values are either functionals on $L^1(\mathscr{A})$ or projections on either $L^\infty(\mathscr{A})$ or $L^1(\mathscr{A})$. Moreover on $\mathscr{A}$, χ is continuous function valued on S, and our next task will be to show how, even on 2^S, χ is continuous function valued, with this function precisely known.

4. $\mathscr{M}$ AND $L^\infty(\mathscr{A})$ FOR C.C.C. ALGEBRAS $\mathscr{A}$: THE RADON-NIKODYM THEOREM

A Boolean algebra $\mathscr{A}$ meets the countable chain condition (c.c.c.) if any pairwise disjoint (p.w.d.) subset of $\mathscr{A}$ is at most countable; disjoint means $ab = 0$. Such an algebra is necessarily complete—$\bigvee_{b\in B} b$ exists in $\mathscr{A}$ for every $B \subset \mathscr{A}$ [15]. It then follows (e.g., see [26, 6.3]) that $L^1(\mathscr{A})$ is precisely Dixmier's normal measures on S, and then from a variety of known results that $L^\infty(\mathscr{A})$ must be $C(S)$. We will reprove this simply and directly, using only two very basic results, and at the same time give for each $\phi \in L^1(\mathscr{A})'$ and $x \in S$, the value $f(x)$ of the representing function $f \in C(S)$; that is, the proof will be more than existential, but more importantly, tells exactly how to realize $L^1(\mathscr{A})'$ for any algebra $\mathscr{A}$ (Section 5).

We make use of two facts: (1) [11] If ν is a measure on $\mathscr{A}$ such that $\nu(b) > 0$ for some $b \in \mathscr{A}$, then $\nu_a > 0$ for some $a \in \mathscr{A}$; indeed $a = \bigvee\{b: \nu_b \geqslant 0\}$. (2) Under 1.1 and with c.c.c. there must exist a strictly positive measure μ on $\mathscr{A}$, written $\mu \gg 0$ and meaning $\mu(a) > 0$ for all $a \neq 0$ [26, 6.5]. The standard example is the measure algebra $\mathscr{A} = \Sigma/\mu^{-1}(0)$ where μ a finite measure on a σ-algebra of sets $\sum$.

THEOREM 4.1. *Let $\mu \gg 0$ on $\mathscr{A}$ and let ν be a measure on $\mathscr{A}$. Let $x \in S$. Then $D_\mu\nu(x) = \lim_{a\to x} \nu(a)/\mu(a)$ exists in $[-\infty, \infty]$.*

Proof. The limit of course means the limit of $\nu(a)/\mu(a)$ over the neighborhood filter of clopen sets a containing x.

Suppose $\overline{\lim}_{a\to x} \nu(a)/\mu(a) > \sigma > \tau > \underline{\lim}_{a\to x} \nu(a)/\mu(a)$. We will produce an uncountable pairwise disjoint subset of $\mathscr{A}$ which will be a contradiction.

First pick a_1' such that $\nu(a_1') > \sigma\mu(a_1')$. By (1) above applied to the measure $(\nu - \sigma\mu)_{a_1'}$ there exist $a_1 \leqslant a_1'$ such that $(\nu - \sigma\mu)_{a_1} > 0$. Now $x \notin a_1$. For if $x \in a_1$, then for all $b \leqslant a_1$, $x \in b$, we would have $\nu(b)/\mu(b) \geqslant \sigma > \tau$ a contradiction. Hence there must be an a_2' such that $x \in a_2'$ and $a_2'a_1 = 0$, and such that $\nu(a_2') > \sigma_\mu(a_2')$. Find a_2 similarly, with $x \notin a_2$, and, since $a_2 \leqslant a_2'$, with $a_2a_1 = 0$.

Let ω_1 be the first uncountable ordinal and suppose we have found $\{a_\alpha\}_{\alpha<\xi}$, $\xi < \omega_1$, such that $\{a_\alpha\}_{\alpha<\xi}$ is p.w.d. and $x \notin a_\alpha$ for all α and $(\nu - \sigma\mu)_{a_\alpha} > 0$.

Then $x \notin \bigvee_{\alpha<\xi} a_\alpha$. For, if $x \in \bigvee_{\alpha<\xi} a_\alpha$, then for all $b \leqslant \bigvee_{\alpha<\xi} a_\alpha$ with $x \in b$ we obtain $b = \bigvee_{\alpha<\xi} ba_\alpha$ and since $\xi < \omega_1$ and $\{a_\alpha\}_{\alpha<\xi}$ are p.w.d., we then obtain $\nu(b) = \sum_{\alpha<\xi} \nu(ba_\alpha) = \sum_{\alpha<\xi} \nu_{a_\alpha}(b) \geqslant \sum_{\alpha<\xi} \sigma\mu_{a_\alpha}(b) = \sigma\mu(b)$ or $\nu(b)/\mu(b) \geqslant \sigma > \tau$ for any such b, the same contradiction as before. Let now $x \in a_\xi$, $a_\xi \cdot \bigvee_{\alpha<\xi} a_\alpha = 0$ and find a_ξ such that $(\nu - \sigma\mu)_{a_\xi} > 0$.

So now, there must exist $\{a_\alpha\}_{\alpha<\omega_1}$, p.w.d., and since each $(\nu - \sigma\mu)_{a_\alpha} > 0$, necessarily all non-zero, a final contradiction to c.c.c.

THEOREM 4.2. *$D_\mu\nu$ is continuous as an extended real-valued function on S.*

Proof. Fix x and suppose $D_\mu\nu(x) > \alpha$. We claim there is an $a \in \mathscr{A}$ such that $x \in a$, and for all $t \in a$, $D_\mu\nu(t) > \alpha$.

For, pick a_1 such that $x \in a \leqslant a_1$ implies $\nu(a)/\mu(a) > \alpha + \epsilon$ for some fixed $\epsilon > 0$, $\epsilon < D_\mu\nu(x) - \alpha$.

Then $(\nu - (\alpha + \epsilon)\,\mu)\,(a_1) > 0$. If $(\nu - (\alpha + \epsilon)\,\mu)_{a_1} \geqslant 0$ we are done. If not, there exists a $b_1 \in \mathscr{A}$, $b_1 \leqslant a_1$ such that $(\nu - (\alpha + \epsilon)\,\mu)_{b_1} < 0$, and by choice of a_1, $x \notin b_1$. So $x \in a_2 - b_1$ and we can repeat the argument, finding $b_2 \leqslant a_2 - b_1$, whence $b_2 b_1 = 0$, such that $(\nu - (\alpha + \epsilon)\,\mu)_{b_2} < 0$.

Suppose we have found $\{b_\delta\}_{\delta<\xi<\omega_1}$, p.w.d., with $(\nu - (\alpha + \epsilon)\,\mu)_{b_\delta} < 0$ and $x \notin a_1 b_\delta$ for all δ, and $b_\delta \leqslant a_1$. Let $b = \bigvee_{\delta<\xi} b_\delta$. Now $x \notin b$. For otherwise $b \leqslant a_1$ and we have $(\nu - (\alpha + \epsilon)\,\mu)\,(b) \geqslant 0$. But $(\nu - (\alpha + \epsilon)\,\mu)\,(b) = \sum_{\delta<\xi} (\nu - (\alpha + \epsilon)\,\mu)\,(b_\delta) < 0$. It again follows that we can find p.w.d. $\{b_\delta\}_{\delta<\omega_1}$ with $(\nu - (\alpha + \epsilon)\,\mu)_{b_\delta} < 0$, and so necessarily all non-zero, a contradiction.

Hence $D_\mu\nu$ is lower semicontinuous at x, and then so also is $D_\mu(-\nu) = -D_\mu\nu$, showing that $D_\mu\nu$ is continuous on S.

Since the possibility $D_\mu\nu(x) = \pm\infty$ exists this does not mean that $D_\mu\nu \in C(S)$. Let $C^\infty(S)^+$ denote the set of continuous functions on S into $[0, \infty]$ which are finite valued on all but a closed nowhere dense set, and give $C^\infty(S)^+$ the pointwise order.

PROPOSITION 4.3. *The correspondence $\hat{\nu} \to D_\mu\nu$ is a linear order isomorphism of $L^1(\mathscr{A})^+$ into $C^\infty(S)^+$.*

Proof. The map is clearly linear and $\hat{\nu} \leqslant \hat{\delta}$ implies $D_\mu\nu \leqslant D_\mu\delta$. Conversely if $D_\mu\nu \leqslant D_\mu\delta$ and yet $(\nu - \delta)\,(a) > 0$ for some a, then there is an a for which $(\nu - \delta - \epsilon\delta)_a > 0$ for $\epsilon > 0$ sufficiently small. But then on the open set a, $D_\mu\nu \geqslant (1 + \epsilon)\, D_\mu\delta$ a contradiction, provided that $D_\mu\nu < \infty$ on all but a nowhere dense set.

For this let $b_n = \bigvee\{a\colon (n\mu - \nu)_a \geqslant 0\}$. Then $b_n \nearrow e$ for, if there is a $b \leqslant e - \bigvee b_n$, then $bb_n = 0$ for all n whence $(n\mu - \nu)\,(b) \leqslant 0$ for all n. For if not, there then exists $c \leqslant b$ such that $(n_0\mu - \nu)_c \geqslant 0$ for some n_0. Whence $c \leqslant b_{n_0}$, yet $c \leqslant e - \bigvee b_n$.

Hence $(n\mu - \nu)\,(b) \leqslant 0$ for all n, again a contradiction, since $\mu(b) > 0$ and $\nu(b) < \infty$.

But now on the clopen set b_n, $D_\mu\nu \leqslant n$ hence $D_\mu\nu < \infty$ on all but $e \backslash \bigcup b_n = \bigvee b_n \backslash \bigcup b_n$, a closed nowhere dense set. This completes the proof since the map is an isomorphism by virtue of the order preservance.

Remark 4.4. The map cannot be onto. Let $\bar{f}(x) = 1/x$ on $(0, 1]$. Then $\bar{f} \wedge n$ has a representation $f_n \in C(S)$ where S is the Stone space of the Lebesgue measure algebra $\mathscr{A}$ of $(0, 1]$. Then $f = \sup f_n \in C^\infty(S)^+$, but cannot be any $D_\mu\nu$

The next result describes the inverse relation $D_\mu\nu \to \hat{\nu}$. Recall that for $f \in C(S)$, $\hat{\mu}_f$ is defined (2.4(a)); indeed $\hat{\mu}_f(g) = \int_S fg\, d\bar{\mu}$ for $g \in \mathscr{S}(\mathscr{A})$ where $\bar{\mu}$ is the Borel extension of $\hat{\mu} \circ \chi$ (3.7).

THEOREM 3.5 (Radon–Nikodym).

(a) $D_\mu \mu_f = f$ *for* $f \in C(S)$.

(b) *If* $f = D_\mu \nu \in C(S)$, *then* $\hat{\nu} = \hat{\mu}_f$.

(c) $\{\hat{\mu}_g : g \in \mathscr{S}(\mathscr{A})\}$ *is* $\| \ \|_1$ *dense in* $L^1(\mathscr{A})$.

(d) *If* $\hat{\nu} \in L^1(\mathscr{A})^+$, $f = D_\mu \nu$ *and* $b_n = \vee\{a : (n\mu - \nu)_a \geqslant 0\}$, *then* $f_n = \chi(b_n) f \in C(S)$ *and in the L-space* $L^1(\mathscr{A})$, $\hat{\nu} = \sup_n \hat{\mu}_{f_n}$.

Proof. (a) Fix x and choose a clopen, $x \in a$, such that $\| \chi(a) f - f(x)\, \chi(a) \| < \epsilon$. It follows that

$$\left| \frac{\mu_f(b)}{\mu(b)} - f(x) \right| < \epsilon \qquad \text{for all } b \leqslant a, \quad x \in b.$$

(b) This is (a) and 4.3.

(c) Consider the sequence $b_n \nearrow e$ in the proof of 4.3. Then (1.2(a)) $\| \hat{\nu}_{b_n} - \hat{\nu} \|_1 \to 0$. Clearly, $D_\mu \nu_{b_n} = \chi(b_n)\, D_\mu \nu$ and $D_\mu \nu \leqslant n$ on b_n. Hence by (b) $\hat{\nu}_{b_n} = \hat{\mu}_{f_n}$, where $f_n = \chi(b_n)\, D_\mu \nu$ and hence $\| \hat{\mu}_{f_n} - \hat{\nu} \|_1 \to 0$. But f_n is in the β (indeed norm) closure of $\mathscr{S}(\mathscr{A})$ completing (c).

(d) Obviously $\hat{\mu}_{f_n} \leqslant \hat{\nu}$ and the argument in (c) completes the proof.

An alternate, and informative argument, is to use a "Lebesgue ladder" argument on the range of $\hat{\nu} \in L^{1^+}$ and obtain a partition of S by clopen sets $a_{n+1} \backslash a_n$ where $a_n = \vee\{c : (\alpha_n \mu - \nu)_c \geqslant 0\}$ and $0 \leqslant \alpha_n \leqslant \alpha_{n+1} \to \infty$ with (say) $| \alpha_{n+1} - \alpha_n | \leqslant \epsilon$ for all n. We thank Jim Roberts for pointing this out.
We turn now to $L^\infty(\mathscr{A})$ where infinity valued derivatives do not occur.

THEOREM 4.6. *Let* $\phi \in L^1(\mathscr{A})' = L^\infty(\mathscr{A})$. *Then*

(a) $\hat{\phi}(x) = \lim_{a \to u} (\phi(\hat{\mu}_a)/\mu(a))$ *exists for all* $x \in S$ *and* $\hat{\phi} \in C(S)$.

(b) *For all* $\hat{\nu} \in L^1(A)$, $\phi(\hat{\nu}) = \langle \hat{\phi}, \hat{\nu} \rangle = \int_S \hat{\phi}\, d\bar{\nu}$ *where* $\bar{\nu} = \hat{\nu} \circ \chi$ *on the Borel subsets of* S (3.7).

(c) *The mapping* $\phi \to \hat{\phi}$ *is a linear, isometric, order, algebra isomorphism of* $L^\infty(A)$ *onto* $C(S)$ *whose inverse is given by* (b); *in particular* $\widehat{\phi\psi}(x) = \hat{\phi}(x)\hat{\psi}(x)$ *for all* $x \in S$.

Proof. (a) By 1.2(a), since ϕ is $\| \ \|_1$-continuous, $\nu(a) = \phi(\bar{\mu}_a)$ is a measure on $\mathscr{A}$. Since ϕ is $\| \ \|_1$-bounded $| \phi(\hat{\mu}_a) | \leqslant \| \phi \| \, \| \hat{\mu}_a \|_1$; whence $\hat{\phi}(x)$ exists by 4.1 for all x and $| \hat{\phi}(x) | \leqslant \| \phi \|$. By 4.2, $\hat{\phi} \in C(S)$.

(b) For $a \in \mathscr{A}$, $\phi(\hat{\mu}_a) = \langle \chi(a), \hat{\mu}_{\hat{\phi}} \rangle$ by 4.5(b). Hence for $g \in \mathscr{S}(\mathscr{A})$, $\phi(\hat{\mu}_g) = \langle g, \hat{\mu}_{\hat{\phi}} \rangle = \langle \hat{\phi}, \hat{\mu}_g \rangle$ since $\hat{\phi} \in C(S)$. Since ϕ is $\| \ \|_1$-continuous, 4.5(c) implies $\phi(\hat{\nu}) = \langle \hat{\phi}, \hat{\nu} \rangle$ for all $\hat{\nu} \in L^1(\mathscr{A})$. The integral form follows from the discussion preceeding 2.3 and from 3.7.

(c) We have already observed that $\|\hat{\phi}\| \leqslant \|\phi\|$. Equality follows as in [11] or [26] from continuity of $\hat{\phi}$. Linear order isomorphism follows from 4.3. The map is onto for, as already observed, $C(S) \subset L^\infty(\mathscr{A})$ in general. Finally $\widehat{\phi\psi}(x) = \hat{\phi}(x)\,\hat{\psi}(x)$ for, using (b) several times,

$$\langle \widehat{\phi\psi}, \hat{\nu}\rangle = \langle \phi\psi, \hat{\nu}\rangle = \langle \phi, \hat{\nu}_{\hat{\psi}}\rangle = \langle \hat{\phi}, \hat{\nu}_{\hat{\psi}}\rangle = \langle \hat{\phi}\hat{\psi}, \hat{\nu}\rangle.$$

It follows then that S is the maximal ideal space of the algebra $L^\infty(\mathscr{A})$ and $\phi \to \hat{\phi}$ is the Gelfand transform as well.

COROLLARY 4.7. *$C(S)_\beta$ is semi-reflexive when $\mathscr{A}$ has* c.c.c.

COROLLARY 4.8. *In the pointwise order, $C(S)$ is a conditionally complete lattice: if $\{f_\alpha\}$ is a bounded increasing net in $C(S)$, then $\vee f_\alpha$ exists in $C(S)$.*

Proof. Apply 2.2 and 4.4 to the functional defined by $\phi(\hat{\nu}) = \sup_\alpha \langle f_\alpha, \hat{\nu}\rangle$ on $L^1(\mathscr{A})^+$.

COROLLARY 4.9. *If $\hat{\mu} \gg 0$ and on $\nu \geqslant 0$ $\mathscr{A}$, then*

$$\lim_{a\to x} \frac{\phi(\hat{\mu}_a)}{\mu(a)} = \lim_{a\to x} \frac{\phi(\hat{\nu}_a)}{\nu(a)} \qquad \textit{for all } x \in S.$$

Proof. Denote these by $\hat{\phi}_\mu$ and $\hat{\phi}_\nu$ respectively. From 4.6(b), $\langle \hat{\phi}_\mu, \delta\rangle = \phi(\delta) = \langle \hat{\phi}_\nu, \delta\rangle$ for all $\delta \in L^1(\mathscr{A})$. Hence $\hat{\phi}_\mu = \hat{\phi}_\nu$ since both are continuous, using 1.1.

Of course, that $C(S)$ is the variation norm dual of the normal measures on such a hypertonian S has been long known and is due to Dixmier. The formula 4.6(a) is implicit in that result. What we will offer now is an extension to σ-algebras without c.c.c., (Section 5), a stronger integral representation than 4.6(b) (Section 6), and a characterization of the measurable sets (here and in Section 5).

To begin the latter we first note that for any $E \subset S$, $\widehat{\chi(E)}$ is 0 or 1 on S. For by 3.3, $\chi(E)^2 = \chi(E)$ and by 4.6(c), $\widehat{\chi(E)} = \widehat{\chi(E)^2} = \widehat{\chi(E)}^2$. Hence $\widehat{\chi(E)} = \chi(a)$ for some $a \in \mathscr{A}$ so that $\chi(\mathscr{A}) = \chi(\mathscr{M}) = \chi(2^S)$; hence $\chi(\mathscr{M})$ produces nothing new in $L^\infty(\mathscr{A})$, a reflection of the semi-reflexivity of the c.c.c. case. To continue, we define an analogue of Lebesgue density on the real line. For $E \subset S$, let $d(E) = \{x \in S\colon \widehat{\chi(E)}\,(x) = 1\}$. Let $\bar{E}(E^0)$ denote the closure (interior) of a set E.

THEOREM 4.10. *$\overline{E}^{\circ} \subset d(E) \subset \bar{E}^0$ for any $E \subset S$.*

Proof. Since $d(E)$ is clopen because $\widehat{\chi(E)}$ is 0 on 1, it suffices to obtain

$E^0 \subset d(E) \subset \bar{E}$. Suppose $x \in E^0$ and pick $a_0 \in \mathscr{A}$ so that $x \in a_0 \subset E^0$. With $\mu \gg 0$ on $\mathscr{A}$,

$$\widehat{\chi(E)}(x) = \lim_{\substack{a \to x \\ a \leqslant a_0}} \frac{\chi(E)\,\hat{\mu}_a}{\mu(a)} = \lim_{\substack{a \to x \\ a \leqslant a_0}} \inf_{U \supset E} \frac{\chi(U)\,\hat{\mu}_a}{\mu(a)}$$

where each U is open. But then $a \leqslant a_0$ implies $a \subset U$ whence $\chi(U)\,\hat{\mu}_a = \mu(a)$ and hence $\widehat{\chi(E)}(x) = 1$. That is, $E^0 \subset d(E)$.

Now suppose $x \notin \bar{E}$, find a_0 such that $x \in a_0$ and $a_0 \cap \bar{E} = \square$. Reasoning as above, and using $U \supset E$ such that $U \cap a_0 = \square$ we obtain $\widehat{\chi(E)}(x) = 0$, or, $x \notin d(E)$.

Corollary 4.11. (a) *If* 0 *is open in* S, *then* $\bar{0}$ *is open,* $d(0) = \bar{0}$ *and* $\widehat{\chi(0)} = \chi_{\bar{0}}$.

(b) *If* C *is closed in* S, *then* C^0 *is closed,* $d(C) = C^0$ *and* $\widehat{\chi(C)} = \chi_{C^0}$.

Proof. Apply 4.10 to the facts that $\bar{0}^0 \subset \bar{0} = \overline{\bar{0}^0}$ and $\bar{C}^0 = C^0 \subset \overline{C^0}$.

Theorem 4.12. *Let* $E \subset S$. *Then*

(a) $\chi(E) = \chi(\bar{E}^0) = \chi(\bar{E})$

(b) $d(E) = \bar{E}^0$

(c) $\widehat{\chi(E)} = \chi_{\bar{E}^0}$ and $\widehat{\chi(X\backslash E)} = 1 - \chi_{\overline{E^0}}$.

Proof. Note first from 4.11 that $\widehat{\chi(0)} = \chi_{\bar{0}} = \chi_{\bar{0}^{\bar{0}}} = \widehat{\chi(\bar{0})}$ whence $\chi(0) = \chi(\bar{0})$; similarly $\chi(C) = \chi(C^0)$. Hence, $\chi(\bar{E}) = \chi(\bar{E}^0)$. Then by definition

$$\chi(E) = \beta\text{-}\lim_{0 \supset E} \chi(0) = \beta\text{-}\lim_{U \supset E} \chi(U) = \chi(\bar{E})$$

for, $0 \supset E$ implies $\bar{0} = U \supset \bar{E}$ and $\chi(\bar{0}) = \chi(0)$ while also, $U \supset \bar{E}$ implies $U \supset E$. This proves (a).

(b) From (a), $d(E) = d(\bar{E}) = \bar{E}^0$ from 4.11(b).

(c) $\widehat{\chi(E)} = \widehat{\chi(\bar{E})} = \chi_{\bar{E}^0}$ again from 4.11(b).

Now, since $\overline{S/E^0} = S/\overline{E^0}$ this also implies the second statement.

This makes our first and principal characterization of measurability apparent.

Theorem 4.13. *These are equivalent*

(a) *E is measurable*

(b) $\bar{E}^0 = \overline{E^0}$

(c) $\overline{E^0} = d(E)$.

Proof. (a) $\Leftrightarrow$ (b). By definition $E \in \mathscr{M}$ iff $\chi(E) + \chi(S\backslash E) = 1$ iff (4.6(c)) $\widehat{\chi(E)} + \widehat{\chi(S\backslash E)} = \hat{1} = 1$ iff (4.12) $\chi_{\bar{E}^0} + (1 - \chi_{\overline{E^0}}) = 1$ iff $\overline{E^0} = \bar{E}^0$.

(b) $\Leftrightarrow$ (c). Apply 4.10 and 4.12(b).

A category measure space is one in which sets of first category and sets of measure zero coincide [20]; $(S, \bar{\mu})$, $\mu \gg 0$ on $\mathscr{A}$ is an example. Our deepest result on measurability is the following.

THEOREM 4.14. *If $\chi(E) = 0$, then E is measurable and nowhere dense. Conversely, if E is nowhere dense, then E is measurable with $\chi(E) = 0$.*

Proof. Since $\chi(E) + \chi(S\backslash E) \geqslant 1$ and $\chi(S\backslash E) \leqslant 1$ for any E, $\chi(E) = 0$ makes E measurable whence by 4.13, $\bar{E}^0 = d(E) = \square$.

Conversely, E nowhere dense makes $\square = \bar{E}^0 = d(E) \supset \overline{E^0}$ whence $\bar{E}^0 = \overline{E^0}$ and E is measurable by 4.13.

A set E in a topological space has the Baire property [20] if $E = 0 \triangle F$ where 0 is open and F is first category.

COROLLARY 4.15. *$E \in \mathscr{M}$ iff E has the Baire property. Indeed $E \in \mathscr{M}$ iff $E \triangle d(E)$ is nowhere dense.*

Proof. Since a first category set F is the countable union of nowhere dense sets, each measurable by 4.13, so is F by 3.6. Since open sets are measurable any $0 \triangle F$ is measurable by 3.6.

Conversely, if $E \in \mathscr{M}$ then $E = d(E) \triangle (E \triangle d(E))$, $d(E)$ is open, and by 4.13, $E \triangle d(E) = E\backslash\bar{E}^0 \cup \overline{E^0}\backslash E$ is first category.

But also each of $E\backslash\bar{E}^0$ and $\overline{E^0}\backslash E$ is nowhere dense, hence of χ measure 0, and hence $\chi(E \triangle d(E)) = 0$ making $E \triangle d(E)$ nowhere dense by 4.14.

Finally, if $E \triangle d(E)$ is nowhere dense, hence measurable, then since $d(E)$ is open, hence measurable, so is $E = d(E) \triangle (E \triangle d(E))$, completing the proof.

So now, each $\hat{\nu} \in L^1(\mathscr{A})$ defines a measure on sets with the Baire property, $\bar{\nu} = \hat{\nu} \circ \chi$, and since S is compact $\bar{\nu}$ has non-empty compact support S_ν.

To conclude this section we note two remarkable properties obtained above. For one, $\chi(2^S) = \chi(\mathscr{M}) = \chi(\mathscr{A})$ as subsets of $L^\infty(\mathscr{A})$ (4.10). For a second, $\widehat{\chi(E)} + \widehat{\chi(S\backslash E)}$ is either 0, 1 or 2 on S (4.10(c)).

5. $\mathscr{M}$ AND $L^\infty(\mathscr{A})_\beta$ FOR σ-COMPLETE $\mathscr{A}$

Let $\hat{\mu} \in L^1(\mathscr{A})^+$ and consider the σ-ideal $\mathscr{I}_\mu = \{a \in \mathscr{A}: \mu(a) = 0\}$ and the resulting Boolean algebra $\mathscr{A}_\mu = \mathscr{A}/\mathscr{I}_\mu$ with $\pi_\mu: \mathscr{A} \to \mathscr{A}_\mu$ the quotient map. It is shown in [14] that the completion $\widehat{S^\tau(R)}$ is a topological projective limit of spaces $L^\infty(\mathscr{A}_\mu)_\beta$. Now $\mathscr{A}_\mu$ is easily seen to be σ-complete with c.c.c. and well-defined strictly positive measure $\mu'(\pi_\mu(a)) = \mu(a)$, and Stone space $S_\mu = S\backslash\cup\{a: \mu(a) = 0\}$, precisely the support of the regular Borel measure $\bar{\mu} = \mu \circ \chi$ of 3.7. Details can be found in [26] or [28]. Since we have described spaces $L^\infty(\mathscr{A}_\mu)_\beta$ we first piece these together into $L^\infty(\mathscr{A})_\beta$; the details are basically those of [14, Sections 7, 8] so we will only describe the structure.

Algebraically there are strong precedents. Gordon [12] described the dual of the space of measures on a σ-algebra algebraically as a space $C(Z)$. More recently Chitescu [3] has done the same for vector-measures, observed that the description is an algebraic projective limit, and noted earlier similar results.

Actually matters are rather straightforward. Let $Y = \cup\{S_\mu: \mu \in L^1(\mathscr{A})^+\}$. Suppose that $x \in Y$, say $x \in S_\mu$. Now $\pi_\mu(a)$ as a clopen set in S_μ is precisely $a \cap S_\mu$ and $\{\pi_\mu(a): x \in a\}$ is the clopen neighborhood base at $x \in S_\mu$. Suppose $\phi \in L^\infty(\mathscr{A})$. Then $\nu'(\pi_\mu(a)) = \phi(\hat{\mu}_a)$ and $\mu'(\pi_\mu(a)) = \mu(a)$ are well-defined on $\mathscr{A}_\mu$, $\mu' \gg 0$, and hence the limit

$$\hat{\phi}(x) = \lim_{\pi_\mu(a)\to x} \frac{\nu'(\pi_\mu(a))}{\mu'(\pi_\mu(a))} = \lim_{a\to x} \frac{\phi(\hat{\mu}_a)}{\mu(a)} \tag{5(a)}$$

exists and is continuous on S_μ in its Stone topology—precisely the relative Stone topology. Note that $|\hat{\phi}(x)| \leqslant \|\phi\|$ as before. The real question is whether ϕ is well-defined on Y and this is the Lebesgue decomposition theorem. Details are similar to [12] (for algebras of sets) and we will only sketch these.

Write $\mu \ll \nu$ if $I_\nu \subset I_\mu$ or equivalently $S_\mu \subset S_\nu$. Suppose $x \in S_\mu$. Then $\hat{\phi}_\mu(x) \equiv \lim_{a\to x} \phi(\hat{\mu}_a)/\mu(a)$ is $\hat{\phi}_\nu(x) \equiv \lim_{a\to x} \phi(\hat{\nu}_a)/\nu(a)$. For, if $\hat{\gamma}' \in L^1(\mathscr{A}_\nu)$ define $\phi'(\hat{\gamma}')$ to be $\phi(\hat{\gamma})$ where $\gamma'(\pi_\nu(a)) = \gamma(a)$. Then ϕ' is well-defined and by 4.6(b) $\phi'(\hat{\gamma}') = \int_{S_\nu} \hat{\phi}_\nu \, d\bar{\gamma}' = \int_S \hat{\phi}_\nu \, d\bar{\gamma}$ so that $\phi(\hat{\mu}_a) = \phi'(\hat{\mu}_a') = \int_S \hat{\phi}_\nu \, d\bar{\mu}_a$. From this follows $\hat{\phi}_\mu(x) = \lim_{a\to x} \phi(\hat{\mu}_a)/\mu(a) = \hat{\phi}_\nu(x)$ by the continuity of $\hat{\phi}_\nu$ on S_ν. Now if $x \in S_\mu \cap S_\nu$ write $\nu = \nu^a + \nu^s$ where $\nu^a(b) = \bar{\nu}(b \cap S_\mu)$. Then $\nu^a \ll \mu$ and $\nu^a \ll \nu$, so $\hat{\phi}_\mu(x) = \hat{\phi}_{\nu^a}(x) = \hat{\phi}_\nu(x)$ as before. Hence

THEOREM 5.1. *If $\phi \in L^\infty(\mathscr{A})$, the formula*

$$\hat{\phi}(x) = \lim_{a\to x} \frac{\phi(\hat{\mu}_a)}{\mu(a)}, \qquad x \in S_\mu$$

yields a well-defined function on $Y = \cup\{S_\mu: \hat{\mu} \in L^1(\mathscr{A})^+\}$ which is continuous in the relative Stone topology on each S_μ and bounded by $\|\phi\|$ on Y.

Hence $L^\infty(\mathscr{A})$ "is" a space of functions on a set. Gordon [12] makes $\hat{\phi}$ continuous by *fiat*, declaring S_μ to be open: recall Ex. 2.5(b)—S_μ is more typically closed nowhere dense in S. But the projective limit technique of [14] lends precision, economy and a naturalness to this and ensuing discussions, and to it we now turn.

With $\mu \ll \nu$ let $\pi_{\mu\nu}: \mathscr{A}_\nu \to \mathscr{A}_\mu$ be the natural map and $\rho_{\mu\nu}: S_\mu \to S_\nu$ the identification of S_μ as a subset of S_ν, and ρ_μ: $S_\mu \to S$ the identity on S_μ. These in turn induce

5(b) $\beta - \beta$ continuous (1.4) linear maps R_μ: $\mathscr{S}(\mathscr{A}) \to \mathscr{S}(\mathscr{A}_\mu)$ by $R_\mu f = f \circ \rho_\mu = f \mid S_\mu$ and corresponding $R_{\mu\nu}$.

5(c) $\| \|_1$-isometries E_μ: $L^1(\mathscr{A}_\mu) \to L^1(\mathscr{A})$ by $E_\mu \hat{\gamma} = \widehat{\gamma \circ \pi_\mu}$ where $\gamma = \hat{\gamma} \circ \chi$ (1.4) and corresponding bonding maps $E_{\mu\nu}$: $L^1(\mathscr{A}_\mu) \to L^1(\mathscr{A}_\nu)$.

5(d) $\beta - \beta$ continuous maps P_μ: $L^\infty(\mathscr{A}) \to L^\infty(\mathscr{A}_\mu)$ and corresponding $P_{\mu\nu}$. Of course $P_\mu = E_\mu'$ and $E_\mu = R_\mu'$, and P_μ is the unique extension of R_μ to the completion $L^\infty(\mathscr{A})$ of $\mathscr{S}(\mathscr{A})_\beta$, whence $P_\mu' = E_\mu$ as well on $L^1(\mathscr{A}_\mu)$ since $L^\infty(\mathscr{A})_\beta' = L^1(\mathscr{A})$. These are the structure maps in the direct and inverse limits taken over the directed family $(L^1(\mathscr{A})^+, \leqslant)$. Note that P_μ and R_μ preserve the order and multiplicative structures as well, the latter because of 3.2 and $P_\mu = R_\mu''$. The next result is found in [14] for rings of sets and we will not reprove it.

THEOREM 5.2. (a) *$L^1(\mathscr{A}) = \varinjlim\{E_{\mu\nu}(L^1(\mathscr{A}_\mu))\}$ as Banach lattices with isometric structure maps E_μ.*

(b) *$L^\infty(\mathscr{A}) = \varprojlim\{P_{\mu\nu}(L^\infty(\mathscr{A}_\mu))\}$ as complete locally convex spaces with β-continuous structure maps P_μ.*

Without topology (b) is the result of Gordon [12] and Chitescu [3]. We will comment on the topological property being claimed for its origins are important. Matters come down to 2.1(b) in the following fashion: (1) β convergence in $L^\infty(\mathscr{A})$ is uniform convergence over β-equicontinuous sets H. (2) By 2.1(b), H is (uniformly) absolutely continuous with respect to a *single* $\hat{\mu} \in L^1(\mathscr{A})^+$. (3) Hence $E_\mu(H)$ is β (i.e., β_μ) equicontinuous in $L^1(\mathscr{A}_\mu)$, whence β_μ-convergence in $L^\infty(\mathscr{A}_\mu)$ is uniform over $E_\mu(H)$ and from this follows the theorem by chasing diagrams.

For the set $Y = \cup\{S_\mu: \mu \in L^1(\mathscr{A})^+\}$ we define a set U to be open if $U \cap S_\mu$ is open in the relative Stone topology on each S_μ; i.e., $x \in U \cap S_\mu$ implies there exists $a \in \mathscr{A}$ such that $x \in a \cap S_\mu \subset U \cap S_\mu$. It is easy to see that $a \cap S_\mu = S_{\mu_a}$, and that this is a topology on Y. Moreover, the collection $\{S_\mu: \hat{\mu} \in L^1(\mathscr{A})^+\}$ is a neighborhood base (the Lebesgue decomposition theorem again!) since $S_\nu \cap S_\mu = S_{\nu^a}$, whence S_ν is open. Alternatively $S_\nu \cap S_\mu = S_{\nu \wedge \mu}$ using the lattice properties of $L^1(\mathscr{A})^+$. (1.1(c)). Furthermore each S_μ being compact in S is also compact in Y since any Y-open cover can be replaced by an open cover of sets $a \cap S_\mu$. Hence,

THEOREM 5.3. (a) *Y is a topological space with compact-open neighborhood base $\{S_\mu: \hat{\mu} \in L^1(\mathscr{A})^+\}$.*

(b) *$Y = \varprojlim\{\rho_{\mu\nu}(S_\mu)\}$ with continuous structure maps ρ_μ: $S_\mu \to Y$ the identity map of S_μ into Y.*

(c) *Y is a dense subset of S.*

The definition of open set and of direct limit systems makes (b) obvious, and (c) follows from the hypothesis of 1.1.

Remark 5.4. In general $Y \neq S$. Suppose $\aleph_1 = 2^{\aleph_0}$ and $\mathscr{A} = 2^{[0,1]}$. Then $\hat{\mu} \in L^1(\mathscr{A})$ iff $\mu = \sum a_n \delta_{x_n}$, $x_n \in [0, 1]$, $a_n \in \mathscr{R}$ and $\sum | a_n | < \infty$. Let X denote $[0, 1]$ with discrete topology. Then $S = \beta X$ and $Y = \cup\{\bar{C}^{\beta X}: C \subset [0, 1]$ is countable$\}$ since $S_\mu = \overline{\{x_n\}}$. Since βX is extremally disconnected, S_μ is clopen in βX since $\{x_n\}$ is open in X. But βX is compact whence if $Y = \beta X$, then $X = \bigcup_{i=1}^n S_{\mu_i} = S_\mu$, $\mu = \sum_1^n \mu_i$, so that $\mu \gg 0$ on $\mathscr{A}$ and $\mathscr{A}$ has c.c.c., a contradiction. Note also that Y is an open subset of S.

Let σ denote the Stone topology on S and $\hat{\sigma}$ the topology defined on Y. Topological statements about subsets of Y are understood to be in terms of $\hat{\sigma}$.

Proposition 5.5. (a) *$\sigma \leqslant \hat{\sigma}$ on Y and $\sigma = \hat{\sigma}$ on each S_μ.*

(b) *The closure of a sequence of compact sets in Y is compact. Any compact open set in Y is a set S_μ.*

(c) *Y is pseudocompact and extremally disconnected.*

Proof. (a) This is clear since if $x \in a \in \mathscr{A}$ and $x \in S_\mu$, then $a \cap S_\mu = S_{\mu_a}$ is a $\hat{\sigma}$ neighborhood of x in Y.

(b) If K is compact, then $K \subset S_\mu$ for some μ for $K \subset \bigcup_{i=1}^n S_{\mu_i} \subset S_\mu$ where $\mu = \sum_{i=1}^n \mu_i$. Suppose K_n is compact and $K_n \subset S_{\mu_n}$. Then $\overline{\bigcup_{n=1}^\infty K_n} \subset S_\mu$ where $\mu = \sum_{n=1}^\infty (1/2^n)(\mu_n / \| \hat{\mu}_n \|)$.

If C is compact-open, then $C = \bigcup_{i=1}^n S_{\mu_i} = S_\mu$, where $\mu = \sum_{i=1}^n \mu_i$.

(c) If f is continuous on Y, and $|f(x_n)| \geqslant n$ then f is unbounded on the compact set $\overline{\cup\{x_n\}}$. Hence Y is pseudocompact. Secondly, if U is open in Y, then $\bar{U} \cap S_\mu = \overline{U \cap S_\mu}$ since S_μ is clopen. But $\sigma = \hat{\sigma}$ on S_μ, S_μ is extremally disconnected (well-known or use 4.11) so $\overline{U \cap S_\mu}$ is open in S_μ whence $\bar{U}$ is open in Y.

In particular Y cannot generally be a P-space since a pseudocompact P-space is finite. We can also note from the proof that a $g \in C(Y)$ which vanishes at infinity has compact suport and hence a natural extension to all of S which is upper semi-continuous on S.

To go on it is a formal consequence of 5.3(b), that

5(e) $C(S) = \varprojlim\{R_{\mu\nu}(C(S_\mu))\}$ with structure maps

$$R_\mu: C(Y) \to C(S_\mu), \qquad R_\mu f = f \mid S_\mu = f \circ \rho_\mu$$

Note that R_μ preserves the additive, multiplicative and order structure.

But Section 4 shows that $L^\infty(\mathscr{A}_\mu) \simeq C(S_\mu)$ and we are ready to combine 5(e) with 5.2(b) to obtain

THEOREM 5.6. *The correspondence $\phi \to \hat{\phi}$ defined in 5.1 is a linear, multiplicative and order isomorphism of $L^\infty(\mathscr{A})$ onto $C(Y)$.*

Proof. Let $D_\mu: L^\infty(\mathscr{A}_\mu) \to C(S_\mu)$ be the linear, multiplicative order isomorphism of 4.6(c). From 5(e) and (5.2)(b) it suffices to show that $R_\mu\hat{\phi} = D_\mu P_\mu \phi$ on S_μ. Of course $(R_\mu\hat{\phi})(x) = \hat{\phi}(x)$ and by 5.1, $\hat{\phi}(x) = \lim_{a\to x} \phi(\hat{\mu}_a)/\mu(a)$. Recalling the notation $\mu'(\pi_\mu(a)) = \mu(a)$ and that $E_\mu\hat{\mu} = \hat{\mu}'$ (5(c)) and that $E_\mu = P'_\mu$ we obtain

$$\begin{aligned} D_\mu P_\mu \phi(x) &= \lim_{\pi_\mu(a)\to x} \frac{(P_\mu\phi)(\hat{\mu}'_{\pi_\mu(a)})}{\mu(\pi_\mu(a))} \\ &= \lim_{a\to x} \frac{\phi(P'_\mu \hat{\mu}'_{\pi_\mu(a)})}{\mu(a)} \\ &= \lim_{a\to x} \frac{\phi(\hat{\mu}_a)}{\mu(a)} = R_\mu\hat{\phi}(x) \end{aligned}$$

since $\{\pi_\mu(a): x \in a\}$ is the clopen neighborhood base at $x \in S_\mu$.

Recall that the topology β can be just as easily defined on $C(S)$ initially ([26]) and $\widehat{C(S)}_\beta = L^\infty(\mathscr{A})$. Also, since S_μ is compact in S any $f \in C(S_\mu)$ has an extension to an $f \in C(S)$. An interesting summation of the topology in 5.2(b), 5(e) and 5.6 is

THEOREM 5.7. *For $\hat{\mu} \in L^1(\mathscr{A})^+$, let $F_\mu = \{f \in C(S): f \mid S_\mu = \hat{\phi} \mid S_\mu\}$. Then $\{F_\mu\}$ is a cauchy filter in $C(S)_\beta$ and $F_x \to^\beta \phi$.*

Proof. For $\mu \leqslant \nu$, $D_\nu P_\nu(F_\mu) = \{\hat{\phi} \mid S_\nu\}$, so in each $C(S_\mu)$ the filter is eventually constant, hence converges, and in $L^\infty(\mathscr{A}_\mu)$ converges to $P_\mu\phi$. Hence it is cauchy and convergent to ϕ.

Combining 5.6 and 5.5(c), $C_b(Y) = C(\beta Y)$ results, and we arrive at the maximal ideal space Z (Section 2) of the B-algebra $L^\infty(\mathscr{A})$: $Z = \beta Y$. This is an appropriate moment to bring in a final projective limit; compare with Ex. 2.5(c) vis-a-vis the Boolean completion $\hat{\mathscr{A}}$ of $\mathscr{A}$.

THEOREM 5.8. *Let $\mathscr{C}\ell(Y)$ denote the complete Boolean algebra of clopen subsets of the extremally disconnected space Y. Then $\mathscr{C}\ell(Y) = \varprojlim\{\pi_{\mu\nu}(\mathscr{A}_\nu)\}$ with Stone space βY and structure maps $\pi_\mu{}^c: \mathscr{C}\ell(Y) \to \mathscr{A}_\mu$ where $\pi_\mu{}^c(C) = C \cap S_\mu$ for $C \in \mathscr{C}\ell(Y)$.*

Proof. Of course $\mathscr{A}_\mu$ is Boolean isomorphic to $\mathscr{C}\ell(S_\mu) = \{a \cap S_\mu: a \in \mathscr{A}\}$. Replace then $\mathscr{A}_\mu$ by $\mathscr{C}\ell(S_\mu)$ with continuous structure maps $\rho_\mu: S_\mu \to Y$. So C clopen in Y implies $\rho_\mu^{-1}(C) \in \mathscr{C}\ell(S_\mu)$. Conversely if $C_\mu \in \mathscr{C}\ell(S_\mu)$, then 5.3(b) implies there exists a unique continuous map $f: Y \to \{0, 1\}$ such that $\chi_{C_\mu} = f \circ \rho_\mu$ for all μ. Let $C = f^{-1}(1)$. Then $C \in \mathscr{C}\ell(Y)$ and $C_\mu = \rho_\mu^{-1}(C)$. The remaining details are clear.

Now we describe the inverse $\hat{\phi} \to \phi$ first in the weak * sense and then (Section 6) in the strong sense. The universal measure will be crucial there and is useful now. Let $\chi_\mu: \mathscr{A}_\mu \to C(S_\mu)$ be as defined: $\chi_\mu(\pi_\mu(a)) = \chi_{a \cap S_\mu}$.

LEMMA 5.9. *For $E \subset S$, $P_\mu \chi(E) = P_\mu(\chi(E \cap S_\mu)) = \chi_\mu(E \cap S_\mu)$. In particular, $P_\mu \chi(S_\mu) = 1 \in L^\infty(\mathscr{A}_\mu)$.*

Proof. If $\hat{\nu}' \in L^1(\mathscr{A}_\mu)$, $E_\mu \hat{\nu}' = \hat{\nu}$, where $\nu \ll \mu$, then $\langle P_\mu \chi(S_\mu), \hat{\nu}' \rangle = \langle \chi(S_\mu), \hat{\nu} \rangle = \lim_{U \supset S_\mu} \langle \chi(U), \hat{\nu} \rangle = \langle 1, \hat{\nu} \rangle$. Now since P_μ is a multiplicative homomorphism, then by 3.6(a) and (d) $P_\mu \chi(E \cap S_\mu) = P_\mu \chi(E)\, P_\mu \chi(S_\mu) = P_\mu \chi(E)$.

Let U be open in S. The net $\{\chi(a): a \subset U\} \to^\beta \chi(U)$ in $L^\infty(\mathscr{A})$ by definition of $\chi(U)$. If $c = b \cap S_\mu \subset U \cap S_\mu$ in A_μ, then there exists $a \in \mathscr{A}$, $a \subset U$ such that $b \cap S_\mu = a \cap S_\mu$. Hence $\{\chi_\mu(a \cap S_\mu): a \subset U\}$ is cofinal in $\{\chi_\mu(c): c \in \mathscr{A}_\mu,\ c \subset U \cap S_\mu\}$. But $P_\mu \chi(a) = \chi_{a \cap S_\mu} = \chi_\mu(a \cap S_\mu)$ and P_μ is β-continuous, so $P_\mu \chi(U) = \chi_\mu(U \cap S_\mu)$. Now for any E, β-continuity of P_μ implies

$$P_\mu \chi(E) = P_\mu \chi(E \cap S_\mu) = \lim_{V \supset E \cap S_\mu} P_\mu \chi(V) = \lim_{V \supset E \cap S_\mu} \chi_\mu(V \cap S_\mu)$$
$$= \chi_\mu(E \cap S_\mu)$$

since an open set W in S_μ contains $E \cap S_\mu$ iff $W = V \cap S_\mu$ for some V open in S; necessarily $V \supset E \cap S_\mu$.

The notation is almost getting in the way of the facts; 5.9 suggests that we regard $\chi_\mu(E \cap S_\mu)$ as simply $\chi(E \cap S_\mu)$, and with 5.6 and the definition of Y, it is correct to identify $L^\infty(\mathscr{A}_\mu)$ with $\{f \in C(Y): f = 0 \text{ on } Y \backslash S_\mu\}$. In [14, Section 7] it is indeed shown that $L^\infty(\mathscr{A}_\mu)$ is a β-complementary subspace of $L^\infty(\mathscr{A})$. However, we can avoid these permissible adoptions without trouble in what remains and do so.

COROLLARY 5.10. *Let $\overline{\nu'} = \nu' \circ \chi_\mu$ and $\bar{\nu} = \hat{\nu} \circ \chi$ be the Borel extensions* (3.7) *of ν' and ν on S_μ and S respectively where $\hat{\nu} = E_\mu \hat{\nu}'$. Then $\bar{\nu}(E) = \overline{\nu'}(E \cap S_\mu)$ for any Borel set $E \subset S$.*

COROLLARY 5.11. (a) *The inverse map $\hat{\phi} \to \phi$ of $C(Y)$ onto $L^\infty(\mathscr{A})$ is given by $\phi(\hat{\nu}) = \int_{S_\nu} \hat{\phi}\, d\bar{\nu}$.*

(b) $\|\phi\| = \|\hat{\phi}\|$.

(c) $\widehat{\chi(E)} \mid S_\mu = \widehat{\chi_\mu(E \cap S_\mu)} = \chi_{\overline{E \cap S_\mu}^0}$.

Proof. (a) By 4.6(b) and 5.8, $(P_\nu \phi)(\nu') = \int_{S_\nu} \hat{\phi}\, d\overline{\nu'} = \int_{S_\nu} \hat{\phi}\, d\bar{\nu}$ and $P_\nu' \hat{\nu}' = \hat{\nu}$. Apply 5.6, 5(e) and 5.2(b).

(b) Already $\|\phi\| \geq \|\hat{\phi}\|$ and the reverse inequality follows from (a), the continuity of $\hat{\phi}$ and the hypothesis of 1.1.

(c) The first equation follows from $P_\mu \chi(E) = \chi_\mu(E \cap S_\mu)$ (5.9) and the isomorphism $D_\mu: L^\infty(\mathscr{A}_\mu) \to C(S_\mu)$. The second equation follows from 4.12(c) and the fact that S_μ is closed in σ and hence in $\hat{\sigma}$.

THEOREM 5.12. *For any $E \subset S$,*

$$\widehat{\chi(E)} = \chi_{\overline{E \cap Y}^0}$$

where the closure and interior are taken in the $\hat{\sigma}$ topology in Y.

Proof. From 5(e) it suffices to show that $\widehat{\chi(E)} \mid S_\mu = \chi_{\overline{E \cap Y}^0} \mid S_\mu$. From 5.11 $\widehat{\chi(E)} \mid S_\mu = \chi_{\overline{E \cap S_\mu}^0}$. Since S_μ is clopen in Y, the conclusion follows.

COROLLARY 5.13. *For any $E \subset S$, $\chi(E) = \chi(\overline{E \cap Y}^0)$.*

Proof. By 5.6 it suffices to show $\widehat{\chi(E)} = \widehat{\chi(\overline{E \cap Y}^0)}$ and for this we only need observe that $(\overline{E \cap Y}^0)^{-0} = \overline{E \cap Y}^0$ since (5.5) Y is extremely disconnected.

Consequently, we can characterize the measurable sets E in S—i.e. those for which $\chi(E) + \chi(S \backslash E) = 1$.

THEOREM 5.14. *These are equivalent for $E \subset S$.*

(a) $E \in \mathscr{M}$

(b) $\overline{E \cap Y}^0 = \overline{(E \cap Y)^0}$ *in the $\hat{\sigma}$ topology on Y.*

(c) $S_{\mu_E} \cap S_{\mu_{S/E}} = \square$ *for all $\hat{\mu} \in L^1(\mathscr{A})^+$.*

(d) $E \cap S_\mu \in \mathscr{M}_\mu$ *for all $\hat{\mu} \in L^1(\mathscr{A})$, where $\mathscr{M}_\mu$ denotes the measurable sets for χ_μ on S_μ.*

Proof. (a) $\Leftrightarrow$ (b). If $E \in \mathscr{M}$, then by 5.13 and 5.6

$$\chi_{\overline{E \cap Y}^0} + \chi_{\overline{S/E \cap Y}^0} = 1 \qquad \text{on} \qquad Y.$$

But $\overline{S \backslash E \cap Y}^0 = Y \cap S \backslash \overline{(E \cap Y)^0} = Y \backslash \overline{(E \cap Y)^0}$. Hence $\overline{E \cap Y}^0 = Y \backslash (Y \backslash \overline{(E \cap Y)^0}) = \overline{(E \cap Y)^0}$.

The converse follows from this as well.

(b) $\Leftrightarrow$ (d) From 4.13, $E \cap S \in \mathscr{M}_\mu$ iff $\overline{E \cap S_\mu}^0 = \overline{(E \cap S_\mu)^0}$ in the $\sigma = \hat{\sigma}$ topology on S_μ. But S_μ is clopen and all together cover Y.

(c) $\Leftrightarrow$ (d) Recall the definition of $\hat{\mu}_E$: $\hat{\mu}_E = \hat{\mu}_{\chi(E)}$. We claim that $S_{\mu_E} = \overline{S_\mu \cap E}^0 = \{x: \chi(E \cap S_\mu)(x) = 1\}$. Let $b \in \mathscr{A}$. Then $\chi(E)(\hat{\mu}_b) = \mu_E(b)$ and $\chi(E \cap S_\mu)(\hat{\mu}_b) = \chi(E)(\hat{\mu}_b) = \chi(E \cap S_\mu)(\hat{\mu}_E)_b$ from 5.8 and $\chi(E)\chi(E \cap S_\mu) = \chi(E \cap S_\mu)$ (3.3).

If $x \in S_{\mu_E}$, then $\widehat{\chi(E \cap S_\mu)}(x) = \lim_{a \to x}[\chi(E \cap S_\mu)(\hat{\mu}_E)_a]/\mu_E(a) = \lim_{a \to x} \mu_E(a)/\mu_E(a) = 1$ whence $S_{\mu_E} \subset \overline{S_\mu \cap E}^0$. If $x \notin S_{\mu_E}$ yet $x \in \overline{S_\mu \cap E}^0 \subset S_\mu$, then there exist b such that $x \in b$ and $\mu_E(b) = 0$. Hence $1 = \widehat{\chi(E \cap S_\mu)}(x) = \lim_{b \geqslant a \to x}[\chi(E \cap S_\mu)(\hat{\mu}_a)]/\mu(a) = \lim_{b \geqslant a \to x} \mu_E(a)/\mu(a) = 0$. Thus $S_{\mu_E} = \overline{S_\mu \cap E}^0$.

Now $S_{\mu_E} \cap S_{\mu_{S \setminus E}} = \{x : [\widehat{\chi(E \cap S_\mu)}\, \widehat{\chi(S \setminus E \cap S_\mu)}](x) = 1\}$ and (c) and 5.12 completes the proof.

Corollary 5.15. *$E \in \mathscr{M}$ iff $\chi(E)\,\chi(F) = \chi(E \cap F)$ for every set $F \subset S$.*

Proof. If $E \in \mathscr{M}$ apply 3.6(d). Suppose $\chi(E)\,\chi(F) = \chi(E \cap F)$ for any F. If $E \notin \mathscr{M}$ the proof of 5.14 shows there exists $x \in Y$ such that

$$1 = \widehat{\chi(E \cap S_\mu)}\,\widehat{\chi(S \setminus E \cap S_\mu)} = \widehat{\chi(E \cap S_\mu)\,\chi(S \setminus E \cap S_\mu)}$$

$$= \widehat{\chi(S_\mu)\,\chi(E)\,\chi(S \setminus E \cap S_\mu)} = \widehat{\chi(S_\mu)\,\chi(\square)} = 0 \quad \text{at} \quad x.$$

Corollary 5.16. *Every open set in Y, hence any union $\cup\{S_\mu : \hat{\mu} \in M \subset L^1(\mathscr{A})^+\}$ is measurable.*

Proof. If V is open then $\bar{V}$ is open since Y is extremally disconnected. Hence $\bar{V}^0 = \bar{V} = \overline{V^0}$.

Theorem 5.17. *If $\chi(E) = 0$, then $E \in \mathscr{M}$ and $E \cap Y$ is nowhere dense. If $E \cap Y$ is nowhere dense, then $E \in \mathscr{M}$ and $\chi(E) = 0$.*

Proof. $\chi(E) = 0$ implies $E \in \mathscr{M}$ just as before. Now $\widehat{\chi(E)} = \chi_{\overline{E \cap Y}^0}$ so $E \cap Y$ is nowhere dense.

Conversely, $E \cap Y$ nowhere dense implies $\overline{(E \cap Y)^0} \subset \overline{E \cap Y}^0$ since Y is extremally disconnected. But the latter is empty. Apply 5.14(b), to obtain $E \in \mathscr{M}$ and $\widehat{\chi(E)} = 0$.

As in Section 4.

Corollary 5.18. *$E \in \mathscr{M}$ iff $E \cap Y$ has the Baire property. Or, $E \in \mathscr{M}$ iff $(E \cap Y) \triangle \overline{(E \cap Y)^0}$ is nowhere dense.*

Proof. Argue as in 4.15 replacing $d(E)$ by $\overline{E \cap Y^0}$.

Corollary 5.19. *$S \setminus Y \in \mathscr{M}$ and $\chi(S \setminus Y) = 0$.*

Hence even for σ-complete $\mathscr{A}$, $(\chi, \mathscr{M})$ is a complete measure space (as is $(\chi, Y \cap \mathscr{M})$ whose measurable sets can be characterized topologically.

We close this section with some applications to vector and normal measures.

For a Boolean algebra $\mathscr{B}$ and locally convex space W let $\eta(\mathscr{B}, W)$ denote the normal measures on $\mathscr{B}$ to W: $G \in \eta(\mathscr{B}, W)$ iff G is a measure and if $b = \bigvee_{\alpha \in \Gamma} b_\alpha$ in $\mathscr{B}$ with $\{b_\alpha\}$ p.w.d., then $G(b) = \lim\{\sum_{\alpha \in \Gamma'} G(b_\alpha)$: Γ' is finite in $\Gamma\}$ in W.

Let $\bar{\beta}$ denote the topological projective limit topology on $C_b(Y)$ obtained from 5(e) and the topologies β_μ defined on $C(S_\mu)$ by $\mathscr{A}_\mu$. Hence since D_μ: $L^\infty(\mathscr{A}_\mu)_{\beta_\mu} \to C(S_\mu)_{\beta_\mu}$, and $D_\mu \phi = \hat{\phi}$ is a topological isomorphism, the map $\phi \to \hat{\phi}$ of 5.6 is a $\beta - \bar{\beta}$ topological isomorphism. The notation $\bar{\beta}$ is consistant with [17] and [26]. The very definition of $\chi(E)$ is virtually a call that χ be normal into $L^\infty(\mathscr{A})_\beta$ and indeed

THEOREM 5.20. *On the complete Boolean algebra $\mathscr{C}\ell(Y)$ of clopen (or regular open) subsets of Y, the measure $\chi: \mathscr{C}\ell(Y) \to L^\infty(\mathscr{A})_\beta$ by $E \to \chi(E)$ is normal. The measure $\bar{\chi}: \mathscr{C}\ell(Y) \to C(Y)_{\bar{\beta}}$ defined by $\bar{\chi}(C) = \chi_C$ is also normal.*

Proof. In $\mathscr{C}\ell(Y)$, $\bigvee C_\alpha = \overline{\cup C_\alpha}$ in Y and $C \vee D = C \cap D$ since Y is extremally disconnected. Hence χ is finitely additive. From 5.2(b), for $\{C_\alpha\}$ p.w.d. $\chi(\bigvee C_\alpha) =^\beta \lim \Sigma\chi(C_\alpha)$ iff $P_\mu\chi(\bigvee C_\alpha) =^{\beta_\mu} \lim P_\mu(\Sigma\chi(C_\alpha))$. But (5.9) $P_\mu\chi(\bigvee C_\alpha) = \chi_\mu((\bigvee C_\alpha) \cap S_\mu)$ and $P_\mu\chi(C_\alpha) = \chi_\mu(C_\alpha \cap S_\mu)$. Now each $C_\alpha \cap S_\mu \in \mathscr{C}\ell(S_\mu)$ and hence $C_\alpha \cap S_\mu = a_\alpha \cap S_\mu$, $a_\alpha \in \mathscr{A}$. Also, in the complete algebra $\mathscr{C}\ell(S_\mu)$, $\bigvee(C_\alpha \cap S_\mu) = (\bigvee C_\alpha) \cap S_\mu$ since $S_\mu \in \mathscr{C}\ell(Y)$. But then $\{C_\alpha \cap S_\mu\}$ is p.w.d. in S_μ, $\mathscr{A}_\mu = \mathscr{C}\ell(S_\mu)$ has c.c.c., so $C_\alpha \cap S_\mu = \square$ for all but countably many α. That is $\bigvee(C_\alpha \cap S_\mu) = \bigvee(C_{\alpha_n} \cap S_\mu)$. But $\chi_\mu: \mathscr{C}\ell(S_\mu) \simeq \mathscr{A}_\mu \to L^\infty(\mathscr{A}_\mu)_{\beta_\mu}$ is a measure (1.1(a)). Since $\phi \to \hat{\phi}$ is a $\beta - \bar{\beta}$ isomorphism and $\bar{\chi}(C) = \chi(C)$, this completes the proof.

THEOREM 5.21. *Let W be a complete locally convex space and $ca(\mathscr{A}, W)$ the countably additive measures on $\mathscr{A}$ into W. Then $ca(\mathscr{A}, W)$ is isomorphic to $\eta(\mathscr{C}\ell(Y), W)$ by $m \to \overline{m} = \tilde{m} \circ \chi$, where $\tilde{m}: L^\infty(\mathscr{A})_\beta \to W$ is the unique continuous linear map defined by m* (1.1(a)) *and χ is as in* 5.20.

Proof. Given m, since $\tilde{m}$ is β-continuous, and $\chi: \mathscr{C}\ell(Y) \to L^\infty(A)_\beta$ is normal, then $\overline{m} = \tilde{m} \circ \chi$ is normal into W. If $\bar{n} \in \eta(\mathscr{C}\ell(Y), W)$, then $\eta(a) = \bar{\eta}(a \cap Y)$ is defined since $a \cap Y \in \mathscr{C}\ell(Y)$, and is in $ca(\mathscr{A}, W)$ since $a_n \nearrow a$ in $\mathscr{A}$ implies $a_n \cap Y \nearrow a \cap Y$ in $\mathscr{C}\ell(Y)$. Since $\bar{\eta} = \tilde{\eta} \circ \chi$ because $\bar{\eta} = \tilde{\eta} \circ \chi$ on the β-dense subset $\mathscr{S}(\mathscr{A})$, this shows that $m \to \overline{m}$ is onto. It is $1 - 1$ since $\overline{m} = \overline{m'}$ implies $\tilde{m} = \tilde{m}'$ on $\mathscr{S}(\mathscr{A})$ which makes $m(a) = \tilde{m}\chi(a) = m'(a)$.

Since $ca(\mathscr{A}, \mathscr{R})$ is $L^1(\mathscr{A})$ (1.1(b)), then by 5.2(a),

COROLLARY 5.22. *$L^1(\mathscr{A})$ is isomorphic to $\eta(\mathscr{C}\ell(Y), \mathscr{R})$. That is, $L^1(\mathscr{A})$ is isomorphic to $\varprojlim \eta(\mathscr{A}_\mu, \mathscr{R}) = \eta(\varprojlim \mathscr{A}_\mu, \mathscr{R})$ since $L^1(\mathscr{A}_\mu) = \eta(\mathscr{A}_\mu, \mathscr{R})$ and $\mathscr{C}\ell(Y) \simeq \varprojlim \mathscr{A}_\mu$* (5.8).

A few concluding remarks are warranted. Of course if $\mathscr{A}$ has c.c.c., $Y = S = S_\mu$ where $\mu \gg 0$. The pair $(\chi, \mathscr{M})$ is very classical, but for its range, as a measure space. We will see in Section 7 that the reduced measure algebra $\mathscr{M}/\chi^{-1}(0)$ is in fact $\mathscr{C}\ell(Y)$ so that the reduced measure $\chi(E \triangle \chi^{-1}(0)) \equiv \chi(\overline{E \cap Y})^0)$ is completely additive (normal). Indeed then χ is strictly positive as a vector measure on $\mathscr{C}\ell(Y)$ into $L^\infty(\mathscr{A})$. Only the non-metrizability of β on $L^\infty(\mathscr{A})$ prevents $\mathscr{C}\ell(Y)$ from having c.c.c. Now we will make $L^\infty(\mathscr{A})$ itself look like a classical L^∞-space, in terms of χ.

6. Inversion and spectral representation

The realization of the Gelfand representation $\phi \to \hat{\phi}$ of $L^\infty(\mathscr{A})$ as $C_b(Y)$ with maximal ideal space $Z = \beta Y$ of the preceeding section (5.6) is a destressingly localized representation. This section attempts a global relationship utilizing the measurable sets $\mathscr{M}$ and extended measure χ. The situation is decidedly analogous to that of classical L^∞-spaces in that once a sufficiently large class of measurable sets is defined, all bounded measurable functions are uniform limits of simple functions. In particular, if we let $\mathscr{S}(\mathscr{M}) = \{\sum_{i=1}^n \alpha_i \chi(E_i)$: $E_i \in \mathscr{M}\} \subset L^\infty(\mathscr{A})$, then (6.2) $L^\infty(\mathscr{A}) = \overline{\mathscr{S}(\mathscr{M})}^{\|\ \|}$ and $\mathscr{S}(\mathscr{M})$ differs from $\mathscr{S}(\mathscr{A})$ by only the inclusion of the two β-limits preceeding 3.2. So matters are like Lebesgue measure on the line with $\mathscr{S}(\mathscr{A})$ playing the role of $\{\sum_{i=1}^n \alpha_i \chi_{(a_i, b_i]}\}$. We begin with

THEOREM 6.1. *Let $\phi \in L^\infty(\mathscr{A})$ with $\phi \geqslant 0$. In the conditionally complete lattice* (2.2) $L^\infty(\mathscr{A})$,

$$\phi = \sup\{s \in \mathscr{S}(\mathscr{M}) : s \leqslant \phi\}.$$

Proof. Let $A = \{y \in Y : \hat{\phi}(y) < \alpha\}$. Then A is open in Y and by 5.16, $A \in \mathscr{M}$.

Let $\epsilon > 0$ and suppose $\|\phi\| \leqslant 1$. Let $E_k = \{y \in Y : k\epsilon < \hat{\phi}(y) \leqslant (k+1)\epsilon\}$ for $k = 1, 2, \ldots, [1/\epsilon] + 1 = n$. Then each $E_k \in \mathscr{M}$ from the above and 3.6. Let $s = \sum_{k=1}^n k\epsilon\, \chi(E_k)$. Now the E_k have p.w.d. interior which have p.w.d. closure and $\sum_{k=1}^n k\epsilon\, \chi_{E_k{}^0} \leqslant \hat{\phi}$ on Y whence $\hat{s} = \sum_{k=1}^n k\epsilon\, \chi_{\overline{E_k{}^0}} \leqslant \hat{\phi}$ so that $s \leqslant \phi$ by 5.6.

Let $\hat{\nu} \in L^1(\mathscr{A})^+$, $\nu(e) \leqslant 1$. Then from 5.11(a) $|\phi(\hat{\nu}) - s(\hat{\nu})| = |\int_{S_\nu} \hat{\phi}\, d\bar{\nu} - \int_{S_\nu} s\, d\bar{\nu}| \leqslant \sum_{k=1}^n \int_{E_k \cap S_\nu} |\hat{\phi} - k\epsilon|\, d\bar{\nu} \leqslant \epsilon$, since $\chi(E_k \triangle \overline{E_k{}^0}) = 0$, where of course $\bar{\nu} = \hat{\nu} \circ \chi$ on $\mathscr{M}$. Consequently, $\|\phi - s\| \leqslant \epsilon$. This proves 6.1 as well as

COROLLARY 6.2. (a) *ϕ is the limit in the norm on $L^\infty(\mathscr{A})$ of the directed family $\{s \in \mathscr{S}(\mathscr{M}) : s \leqslant \phi\}$.*

(b) *ϕ is the norm limit of the directed family $\{\sum_{i=1}^n \alpha_i \chi(b_i)$: b_i is clopen in Y and $\sum_{i=1}^n \alpha_i \chi_{b_i} \leqslant \hat{\phi}\}$.*

Proof. It suffices to observe that if $\phi \geqslant s' \geqslant s$ in $L^\infty(\mathscr{A})$, then $\| \phi - s' \| \leqslant \| \phi - s \|$, and to note that each $\overline{E_k{}^0}$ above is clopen in Y.

Remark. In example 2.5(c), $\{s \in \mathscr{S}(\mathscr{A}): s \leqslant \phi\} = \{0\}$. Thus one must use $\mathscr{S}(\mathscr{M})$ to obtain $L^\infty(\mathscr{A})$.

Both 6.1 and 6.2 strongly suggest a formula $\phi = \int \hat{\phi}\, d\chi$ as the inversion of the map $\phi \to \hat{\phi}$ of $L^\infty(\mathscr{A})$ onto $C_b(Y)$. Indeed we already have this as a weak * integral, globally over S in 4.6 and locally over sets S_ν in 5.11. We seek a global vector-valued integral. Existing vector integral structures known to us, [1], [10], are not applicable because $\chi: \mathscr{M} \to L^\infty(\mathscr{A})$ is not a measure in the norm, but only in β, this of course the fundamental reason for introducing β on $\mathscr{S}(\mathscr{A})$.

A definition of integral appropriate to the inversion of $\phi \to \hat{\phi}$ and an allied spectral representation is given below. Despite its special appearance we will later at least indicate that it has a more universal meaning closely tied to the fact that χ is itself universal for measures.

Let X be a set and $\Sigma \subset 2^X$ a σ-algebra of sets. A measure m on X into $L^\infty(\mathscr{A})$ is a finitely additive map $m: \Sigma \to L^\infty(\mathscr{A})$ such that $E \subset F$ implies $m(E) \leqslant m(F)$ and $E_i \nearrow E$ in Σ implies $m(E_i) \to^\beta m(E)$; that is, converges uniformly on weak (= weak *) compacta in $L^1(\mathscr{A})^+$.

Let $B(X, \Sigma) = \{f: X \to R: f \text{ is bounded and } \Sigma\text{-measurable}\}$ and $\mathscr{S}(X, \Sigma) = \{s \in B(X, \Sigma): s = \sum_{i=1}^n \alpha_i \chi_{A_i}\}$. Define $\int_X dm: \mathscr{S}(X, \Sigma) \to L^\infty(\mathscr{A})$ by $\int_X s\, dm = \sum_{i=1}^n \alpha_i m(A_i)$ where $s = \sum_{i=1}^n \alpha_i \chi_{A_i}$, $\{A_i\}$ p.w.d. As usual $\int_X s\, dm$ is well-defined. Now let $f \in B(X, \Sigma), f \geqslant 0$. Then $\{s \in \mathscr{S}(X, \Sigma): 0 \leqslant s \leqslant f\}$ is directed upward and hence so is $\{\int_X s\, dm: 0 \leqslant s \leqslant f\}$ directed upward in $L^\infty(\mathscr{A})$. Moreover, $s \leqslant f$ implies $\int_{X_+} s\, dm \leqslant \| f \|\, m(X) \in L^\infty(\mathscr{A})$ and so now by 2.2 we can define $\int_X dm: B(X, \Sigma) \to L^\infty(\mathscr{A})$ by

$$\int_X f\, dm = \sup \left\{\int_X s\, dm: 0 \leqslant s \leqslant f\right\} = \beta\text{-}\lim_{s \nearrow f} \int_X s\, dm$$

in $L^\infty(\mathscr{A})$, where $s \nearrow f$ pointwise on X. For any f, write $f = f^+ - f^-$, $f^\pm \in B(X, \Sigma)$ and non-negative and define $\int_X f\, dm = \int_X f^+\, dm - \text{r}\ f^-\, dm$.

We will not investigate this integral of itself, but will rather point out a few principal properties and its connection with the Gelfand representation $\phi \to \hat{\phi}$.

THEOREM 6.3. (a) *A finitely additive increasing $m: \Sigma \to L^\infty(\mathscr{A})$ is a measure in $L^\infty(\mathscr{A})$ iff m is $\sigma(L^\infty, L^1)$ countably additive.*

(b) *For any $\hat{\mu} \in L^1(\mathscr{A})$,*

$$\left\langle \int_X f\, dm, \hat{\mu} \right\rangle = \int_X f\, d(m \circ \hat{\mu}) \qquad \textit{where} \qquad (m \circ \hat{\mu})(E) = \langle m(E), \hat{\mu} \rangle$$

the integral on the right being the usual Lebesgue integral of f by the measure $m \circ \hat{\mu}: \Sigma \to R$.

(c) *The equation in* (b) *uniquely defines* $\int_X f\,dm$ *in* $L^\infty(\mathscr{A})$.

(d) $k_f(E) = \int_X f\chi_E\,dm$ *is a measure on* Σ *in* $L^\infty(\mathscr{A})$ *and* $\{k_s\colon 0 \leqslant s \leqslant f\}$ *is uniformly countably additive.*

Proof. It suffices to consider only $f \in B(X, \Sigma)$, $f \geqslant 0$. (a) If m is $\sigma(L^\infty, L^1)$ countably additive, then $m(E_i) \nearrow m(E)$ pointwise on weak = weak * compacta in $L^1(\mathscr{A})^+$, hence uniformly by Dini's theorem, hence in β.

(b) This is apparent from 2.2, $\hat{\mu}$ β-continuity, the definition of $\int_X f\,dm$, and of $\int_X f\,d\nu$ for the real measure $\nu = m \circ \hat{\mu}$.

(c) Suppose $\phi \in L^\infty(\mathscr{A})$ and for all $\hat{\mu} \in L^1(\mathscr{A})$, $\langle \phi, \hat{\mu}\rangle = \int_X f\,d(m \circ \hat{\mu})$. If $0 \leqslant s \leqslant f$, then $\hat{\mu} \geqslant 0$ implies $\langle \phi, \hat{\mu}\rangle \geqslant \int_X s\,d(m \circ \hat{\mu}) = \langle \Sigma\alpha_i m(E_i), \hat{\mu}\rangle$ so $\phi \geqslant \{\int_X s\,dm\colon 0 \leqslant s \leqslant f\}$ whence $\phi \geqslant \int_X f\,dm$. That one cannot have $\phi(\hat{\mu}) > \langle \int_X f\,dm, \hat{\mu}\rangle$ for some $\hat{\mu} \in L^1(\mathscr{A})^+$ now follows from (b).

(d) Let $\hat{\mu} \in L^1(\mathscr{A})^+$. Then

$$\langle k_f(E), \hat{\mu}\rangle = \int f\chi_E\,d(m \circ \hat{\mu}) = \int_E f\,d(m \circ \hat{\mu})$$

which is of course c.a. Hence k_f is $\sigma(L^\infty, L^1)$ countably additive, but being increasing, necessarily β-c.a. by 2.2. The second claim follows from this and the observation that

$$\langle k_s(E \backslash E_i), \hat{\mu}\rangle \leqslant \int_{E/E_i} f\,d(m \circ \hat{\mu})$$

from (b) for all $s \leqslant f$.

To continue, we apply this integral to $(Y, \mathscr{M} \cap Y)$ where $\mathscr{M}$ is the σ-algebra of subsets of S of χ-measurable sets (3.5) and $\mathscr{M} \cap Y = \{E \cap Y\colon E \in \mathscr{M}\}$. Note from 5.16, 5.19 and 3.6(b), that $Y \in \mathscr{M}$, that $\chi(E \cap Y) = \chi(E)$ for all $E \in \mathscr{M}$, and that $\chi\colon \mathscr{M} \cap Y \to L^\infty(\mathscr{A})$ is a measure in $L^\infty(\mathscr{A})$.

THEOREM 6.4. (a) *If* $\phi \in L^\infty(\mathscr{A})$, *then* $\phi = \int_Y \hat{\phi}\,d\chi$.

(b) *If* $f \in B(Y, \mathscr{M} \cap Y)$ *and we define* $\phi = \int_Y f\,d\chi$, *then* $f = \hat{\phi}$ *except on a set of* χ *measure zero in* Y. *Hence any measurable function on* Y *is continuous except at points of a nowhere dense set.*

Proof. By definition, $\int_Y \hat{\phi}\,d\chi = \sup\{\sum_{i=1}^n \alpha_i\chi(E_i)\colon \sum_{i=1}^n \alpha_i\chi_{E_i} \leqslant \hat{\phi}$ on Y, $E_i \in \mathscr{M} \cap Y$ and p.w.d.$\}$. As noted in 6.1 $\Sigma_i\alpha\chi_{E_i} \leqslant \hat{\phi}$ implies $\Sigma\alpha_i\chi(E_i) \leqslant \phi$ and conversely, if $s \in \mathscr{S}(\mathscr{M})$ with $s \leqslant \phi$, then $\hat{s} \in \mathscr{S}(Y, \mathscr{M} \cap Y)$ and $\int_Y \hat{s}\,d\chi = s$, which, with 6.1, proves (a). (Alternatively, apply 6.3(b) and (c) to 5.11(a)).

(b) For $f \in B(Y, \mathscr{M} \cap Y)$ we have already noted that $\int_Y f\,d\chi$ exists in $L^\infty(\mathscr{A})$. Denote it by ϕ. By (a) above $\int_Y f\,d\chi = \int_Y \hat{\phi}\,d\chi$ and by 6.3(b), this yields $\int_Y f\,d\tilde{u} = \int_Y \phi\,d\tilde{u}$ for all $\hat{\mu} \in L^1(\mathscr{A})$ where $\tilde{u} = \hat{\mu} \circ \chi \mid \mathscr{M} \cap Y$. Let $E =$

$\{x \in Y: \hat{\phi}(x) \geqslant f(x) + 1/n\}$. Then $E \in \mathscr{M} \cap Y$ and if $\chi(E) \neq 0$ then $\bar{E}^0 \neq \square$. But since $E \in \mathscr{M}$, $\bar{E}^0 = \overline{E^0}$ and so there exists $y \in Y$ such that $y \in E^0$. Hence there exists $\hat{\mu} \in L^1(\mathscr{A})^+$ such that $\hat{\phi} - f \geqslant 1/n$ on S_μ which implies $\int_Y (\hat{\phi} - f)\, d\tilde{u} \geqslant 1/n$ a contradiction. From this, $\{x \in Y: \hat{\phi} \neq f\}$ must be first category in Y, hence of χ measure zero by 5.17.

Remark. (1) Define $\hat{\phi}_0: S \to \mathscr{R}$ by $\hat{\phi}_0 = \hat{\phi}$ on Y and 0 on $S \backslash Y$. Since $S \backslash Y \in \mathscr{M}$, then $\hat{\phi}_0 \in B(S, \mathscr{M})$ and it follows that $\phi = \int_S \hat{\phi}_0\, d\chi$ since $\chi: \mathscr{M} \to L^\infty(\mathscr{A})$ is a measure as well. Of course $\hat{\phi}_0$ can be defined in any way we please on $S \backslash Y$; see 7.4 for a related result.

(2) In the Aren's multiplication in $L^\infty(\mathscr{A})$, $\widehat{\phi\psi} = \hat{\phi}\hat{\psi}$ whence

$$\int \hat{\phi}\, d\chi \cdot \int \hat{\psi}\, d\chi = \int \hat{\phi}\hat{\psi}\, d\chi = \int \widehat{\phi\psi}\, d\chi.$$

We continue with a spectral integral. Fix $\phi \in L^\infty(\mathscr{A})$ and let $\mathscr{B}_0(\mathscr{R})$ denote the Borel subsets of $\mathscr{R}$. Since $\hat{\phi}$ is continuous on Y and $\mathscr{M} \cap Y$ contains all open sets in Y (5.16), $P_\phi(E) = \chi(\hat{\phi}^{-1}(E))$ exists in $L^\infty(\mathscr{A})$ for all $E \in \mathscr{B}_0(\mathscr{R})$. We write $P = P_\phi$ in the next theorem.

PROPOSITION 6.5. (a) *$P: \mathscr{B}_0(\mathscr{R}) \to L^\infty(\mathscr{A})$ is a measure in $L^\infty(\mathscr{A})$ such that $P(\square) = 0$, $P(\mathscr{R}) = 1$ and $P(E \cap F) = P(E)\, P(F)$ for all $E, F \in \mathscr{B}_0(\mathscr{R})$.*

(b) *$P(E)^2 = P(E)$ and $P(E)\, P(F) = P(E)$ for all $E, F \in \mathscr{B}_0(\mathscr{R})$ with $E \subset F$.*

(c) *$P(\mathscr{R} \backslash \sigma(\phi)) = 0$ where $\sigma(\phi) = \hat{\phi}(Y)$ is the spectrum of ϕ in the algebra $L^\infty(\mathscr{A})$.*

Proof. (a) Since χ is additive on disjoint sets so is P. Let $G, H \in \mathscr{M}$. Then $\chi(G \cap H) = \chi(G)\, \chi(H)$ by 3.6 and $\hat{\phi}^{-1}(E \cap F) = \hat{\phi}^{-1}(E) \cap \hat{\phi}^{-1}(F)$, whence $P(E)\, P(F) = P(E \cap F)$. For any $\hat{\mu} \in L^1(\mathscr{A})$, $\langle P(E), \hat{\mu} \rangle = \langle \chi(\hat{\phi}^{-1}(E)), \hat{\mu} \rangle = \tilde{u}(\hat{\phi}^{-1}(E))$ (where $\tilde{u} = \hat{\mu} \circ \chi \mid \mathscr{M} \cap Y$) is a measure on $\mathscr{R}$. By 6.3(a) $P: \mathscr{B}_0(\mathscr{R}) \to L^\infty(\mathscr{A})$ is a measure.

(b) This is a corollary of the multiplication property in (a).

(c) Of course, $P(R \backslash \hat{\phi}(Y)) = 0$. But $\sigma(\phi) = \hat{\phi}(Y)$ for if $\lambda \notin \hat{\phi}(Y)$, then $1/(\hat{\phi} - \lambda)$ is continuous and hence bounded on Y because Y is pseudocompact (5.5), whence $\sigma(\phi) \subset \hat{\phi}(Y) \subset \sigma(\phi)$ by the isomorphism of $L^\infty(\mathscr{A})$ onto $C(Y)$.

The next result is just a change of variable in 6.4(a) and can be regarded as closest possible description of an arbitrary ϕ as a simple function over $\mathscr{M}$. For the *unbounded* function $f(\lambda) = \lambda$ on $\mathscr{R}$ define $\int_{\mathscr{R}} \lambda\, dP_\phi \equiv \int_{\mathscr{R}} \lambda \chi_{[-\|\phi\|, \|\phi\|]}\, dP_\phi$.

THEOREM 6.6. $\phi = \int_{\mathscr{R}} \lambda\, dP_\phi$.

Proof. We can assume $\phi \geqslant 0$. Let $X = [-\|\phi\|, \|\phi\|]$, $\Sigma = \mathscr{B}_0(X)$. From

6.4, $P_\phi: \Sigma \to L^\infty(\mathscr{A})$ is a measure in $L^\infty(\mathscr{A})$. The proof now follows readily from the definitions and proofs of 6.1 and 6.4 if we only notice that for

$$\hat{s}_1 = \sum_{k=1}^{n} k\epsilon\chi_{(k\epsilon,(k+1)\epsilon]} \in \mathscr{S}(X, \Sigma)$$

and

$$\hat{s}_2 = \sum_{k=1}^{n} k\epsilon\chi_{E_k} \in \mathscr{S}(Y, \mathscr{M} \cap Y)$$

where

$$E_k = \hat{\phi}^{-1}(k\epsilon, (k+1)\epsilon]$$

one has $s_1 = \int_{[0,\|\phi\|]} \hat{s}_1 \, dP_\phi = \int_Y \hat{s}_2 \, d\chi = \sum_{i=1}^n k \epsilon \chi(E_k) = s_2$ where $n = [\|\phi\|/\epsilon] + 1$. This completes the proof.

Remark. From 6.3(b), $\phi(\hat{\mu}) = \int \lambda \, dP_\phi\hat{\mu}$ for all $\hat{\mu} \in L^1(\mathscr{A})$ where $P_\phi\hat{\mu}(E) = \hat{\mu}(\chi(\hat{\phi}^{-1}(E)) = \bar{\mu}(\hat{\phi}^{-1}(E))$ is a real measure on the Borel subsets of $\mathscr{R}$ and of compact support.

Let us call a measure $Q: \mathscr{B}_0(\mathscr{R}) \to L^\infty(\mathscr{A})$ which satisfies 6.5(a) and vanishes outside some compact set in $\mathscr{R}$ a *spectral measure*. It follows that $Q(E)^2 = Q(E)$ for all E and hence $\widehat{Q(E)^2} = \widehat{Q(E)}$ whence $\| Q(E)\|$ is 0 or 1. Set $\phi = \int_{\mathscr{R}} \lambda \, dQ$ If p is a polynomial one can prove $p(\phi) = \int_{\mathscr{R}} p(\lambda) \, dQ$ (see Remark (2) following 6.4) and from this and the Stone–Weierstrass theorem on spaces $C[a, b]$ that the representation 6.6 of $L^\infty(\mathscr{A})$ by spectral measures is unique. One can alternatively obtain the representation in terms of the distribution function $F(\lambda) = F_\phi(\lambda) = \chi(\hat{\phi}^{-1}(-\infty, \lambda])$ and a Stieltjes integral $\phi = \int_{-\|\phi\|}^{\|\phi\|} \lambda \, dF_\phi$ where $F: \mathscr{R} \to L^\infty(\mathscr{A})$ is increasing, right-continuous, $F(\lambda) F(\mu) = F(\mu)$ for all $\lambda \leqslant \mu$ and $\lim_{\lambda \to -\infty} F(\lambda) = 0$, $\lim_{\lambda \to +\infty} F(\lambda) = 1 = \chi(e)$. Using the order on $L^\infty(\mathscr{A})$, F induces the spectral measure P on $\mathscr{B}_0(R)$ in the usual way.

Note that in both representations 6.4 and 6.6 the measures are valued in the idempotents in $L^\infty(\mathscr{A})$. In Section 7 we characterize these in a number of ways.

To close this section we want to indicate that the integration form used here plays a larger role in the theory of integration of real functions by vector measures than their application to the representation here indicates.

Example 6.7. Indeed the integral defined preceeding 6.3 has as a special case the canonical imbedding of bounded (or essentially bounded) measurable functions as continuous functions on the Stone space of the relevant Boolean measure algebras. To see this, let $\Sigma \subset 2^X$ be a σ-field and let $\mathscr{A}$ denote Σ as a Boolean algebra. Define $\bar{\chi}: \Sigma \to L^\infty(\mathscr{A})$ by $\bar{\chi}(A) = \chi(a)$ where $a \in \mathscr{A}$ is $A \in \Sigma$. If $\bar{f} \in B(X, \Sigma)$ it then follows that $f = \int_X \bar{f} \, d\bar{\chi} \in C(S)$ and is the unique representative of $\bar{f}$ in $C(S)$. This is because $\int_X d\bar{\chi}: \mathscr{S}(X, \Sigma) \to \mathscr{S}(\mathscr{A})$ is an isometry onto. In the case of essentially bounded functions mod a σ-ideal $I \subset \Sigma$ we write $A = \Sigma/I$ and $\bar{\chi}(A) = \chi[A]$ where $[A] = A \triangle I$.

But more is true, for just as χ is a universal measure, $\int_X d\bar{\chi}$ is universal in the category *CLCS* of complete locally convex spaces and continuous linear maps. The next definition covers all integrals of real functions by vector measures known to the authors.

DEFINITION. An integral on $B(X, \Sigma)$ into $W \in CLCS$ is a map $I\colon B(X, \Sigma) \to W$ such that

(a) $m(E) = I(\chi_E)$ is a measure on Σ into W.

(b) For all $f \in B(X, \Sigma)$ and $x' \in W'$,

$$\langle I(f), x' \rangle = \int_X f\, dx'm.$$

It follows of course that I is linear.

THEOREM 6.8. *Let $\Sigma \subset 2^X$ be a σ-algebra of sets. Let $\mathscr{A}$ denote Σ as a Boolean algebra and let $\bar{\chi}\colon \Sigma \to \mathscr{S}(\mathscr{A})$ be defined by $\bar{\chi}(E) = \chi(E)$, $E \in \Sigma = \mathscr{A}$.*

(a) *If $I\colon B(X, \Sigma) \to W$ is an integral, then $I = \tilde{m} \circ \int_X d\bar{\chi}$ where $\tilde{m}\colon L^\infty(\mathscr{A})_B \to W$ is the unique continuous linear map such that $m = \tilde{m} \circ \chi$ on $\mathscr{A}$ of* 1.1(a).

(b) *If $m\colon \Sigma \to W$ is a measure, then there exists an integral $I\colon B(X, \Sigma) \to W$ such that $m(E) = I(\chi_E)$ for all $E \in \Sigma$.*

Proof. (a) Define $m\colon \mathscr{A} \to W$ by $m(a) = I(\chi_A)$ where $a \in A$ in $\Sigma = \mathscr{A}$. It follows from (a) in the definition that m is a measure, whence by 1.1(a) $m = \tilde{m} \circ \chi$, $\tilde{m}\colon \mathscr{S}(\mathscr{A})_B \to W$, $\tilde{m}$ unique. Now $\tilde{m}$ has a unique linear continuous extension $\tilde{m}$ on the completion $L^\infty(\mathscr{A})$ into W. We claim that $I(f) = \tilde{m}(\int_X f\, d\bar{\chi})$ for all $f \in B(X, \Sigma)$. Let $x' \in W'$, and let $\tilde{m}'\colon W' \to \mathscr{S}(\mathscr{A})'_B = L^1(\mathscr{A})$ denote the adjoint of $\tilde{m}$. Now $\langle \tilde{m}(\int_X f\, d\bar{\chi}), x' \rangle = \langle \int_X f\, d\bar{\chi}, \tilde{m}'x' \rangle = \int_X f\, d\tilde{m}'x' \circ \bar{\chi}$ by 6.3(b). But, $(\tilde{m}'x' \circ \bar{\chi})(E) = \tilde{m}'x'(\chi(E)) = x'(\tilde{m}\chi(E)) = x'(m(E))$ whence $\int_X f\, d\tilde{m}'x' \circ \bar{\chi} = \int_X f\, dx'm = \langle I(f), x' \rangle$ completing the proof.

(b) If $m\colon \Sigma \to W$ is a measure, then $\mu(a) = m(A)$ where $a = A$ is a measure $\mu\colon \mathscr{A} \to W$ such that $\mu = \tilde{\mu} \circ \chi$ where $\tilde{\mu}\colon \mathscr{S}(\mathscr{A})_B \to W$ is linear, continuous with a unique linear continuous extension to $L^\infty(\mathscr{A})_B$ into W. Define $I(f) = \tilde{\mu}(\int_X f\, d\bar{\chi})$ for $f \in B(X, \Sigma)$; as noted, the integral $\int_X f\, d\bar{\chi}$ always exists in $L^\infty(\mathscr{A})$. Then $I(\chi_E) = \tilde{\mu}(\bar{\chi}(E)) = \tilde{\mu}(\chi(E)) = \mu(E) = m(E)$ and hence $\langle I(f), x' \rangle = \int_X f\, dx'm$ (6.3(b)) for all $f \in B(X, \Sigma)$ and $x' \in W'$, completing the proof.

This in a sense brings us full circle to the link between [14] and [26] and the proposition of Section 1. If Σ is a σ-algebra of sets in X, let $\bar{\chi}\colon \Sigma \to \mathscr{S}(\mathscr{A})$

be $\tilde{\chi}(E) = \chi(E)$ where we regard Σ as a Boolean algebra $\mathscr{A}$. Then $\int_X d\tilde{\chi}: \mathscr{S}(\Sigma) \to \mathscr{S}(\mathscr{A})$ is precisely the algebra isomorphism linking [14] and [26]. If we give $\mathscr{S}(\Sigma)$ the topology τ of [14], then $\int_X d\tilde{\chi}$ becomes a topological isomorphism whose extension to $\widehat{\mathscr{S}^\tau(\Sigma)}$ is a topological isomorphism onto $L^\infty(\mathscr{A})_\beta$.

7. The spectral algebra $\mathscr{M}/\chi^{-1}(0)$

Previous sections establish the set $\chi(\mathscr{M}) \subset L^\infty(\mathscr{A})$ as the basic descriptive agent for the dual of $L^1(\mathscr{A})$, by uniform closure of its linear span. This section establishes the isomorphism of $\chi(\mathscr{M})$ as a set and as a Boolean algebra with a number of other equally basic elements naturally carried by the structures L^1 and L^∞ under study. Specifically we will establish the Boolean identity of the following:

$$\hat{\mathscr{M}} = \mathscr{M}/\chi^{-1}(0)$$
$$I^\infty(\mathscr{A}) = \{\phi \in L^\infty(\mathscr{A}) : \phi^2 = \phi\}$$
$$\mathscr{C}\ell(Y) = \{E \subset Y : E \text{ is clopen}\}$$
$$\mathscr{E} = \{\phi \in L^\infty(\mathscr{A}) : \phi \text{ is an extreme point of } \{\psi \in L^\infty(\mathscr{A}) : 0 \leqslant \psi \leqslant 1 = \chi(e)\}\}$$
$$\mathscr{P}(\mathscr{A}) = \{P : L^1(\mathscr{A}) \to L^1(\mathscr{A}) : P^2 = P \text{ and } \| P\hat{\mu} \| + \|(I-P)\hat{\mu}\| = \|\hat{\mu}\| \text{ for all } \hat{\mu} \in L^1(\mathscr{A})\}$$

when the Boolean operations are suitably defined.

The last class $\mathscr{P}$ is important for it links our work precisely to that of Cunningham [8], who established $\mathscr{P}$ (whose elements Cunningham called L-projections) as the basic descriptive agent in the representation of L-spaces (such as $L^1(\mathscr{A})$) "which is natural and, as far as possible, unique". As Cunningham points out the Kakutani representation of L-spaces is not a unique one.

The identification of the above classes as complete Boolean algebras is considerably simplified by appeal to the following easily proven lemma.

Lemma 7.1. *Let $\mathscr{A}$ be a complete Boolean algebra, $\mathscr{B}$ a Boolean algebra and $\rho: \mathscr{A} \to \mathscr{B}$ a Boolean isomorpism of $\mathscr{A}$ onto $\mathscr{B}$. Then $\mathscr{B}$ is complete and ρ preserves infinite operations. That is, $\rho(\vee E) = \vee \rho(E)$ for any $E \subset \mathscr{A}$.*

We shall say that $\mathscr{B}$ is completely isomorphic to $\mathscr{A}$; and write $\mathscr{B} \simeq \mathscr{A}$.

Thus by isomorphism we mean a $1-1$ onto homomorphism under finite operations. To establish then the identity of the five classes above as complete Boolean algebras we need only establish that one is complete and the remaining isomorphic onto it under suitable operations. A complete one is the previously established algebra $\mathscr{C}\ell(Y)$ of subsets of the extremally disconnected space Y. As a further note it suffices to determine isomorphism in either the ring operations, $+$ and $\cdot$, or the (consistant with $+$ and $\cdot$) lattice operations $\vee$ and $\wedge$ and we use these alternatively.

7.2. $I^\infty(\mathscr{A})$ *and* $\mathscr{C}\ell(Y)$

For $f \in C(Y)$ let $\sigma(f) = \{f \neq 0\}$. Then σ is clearly an isomorphism, under pointwise multiplication and $f \oplus g \equiv f + g - 2fg$, of the idempotents $I(Y)$ in $C(Y)$ onto $\mathscr{C}\ell(Y)$ under $\cap$ and $\triangle$. If we then let D denote the isomorphism $\phi \to \hat{\phi}$ of $L^\infty(\mathscr{A})$ onto $C(Y)$ (5.6), then $\rho = \sigma \circ D$ establishes the desired isomorphism of $I^\infty(\mathscr{A})$ onto $\mathscr{C}\ell(Y)$, where in $I^\infty(\mathscr{A})$, $\phi \oplus \psi \equiv \phi + \psi - 2\phi\psi$. Since both σ and D are order preserving in the natural orders, this also establishes that $\oplus$ is consistent with the order structure on $I^\infty(\mathscr{A})$ inherited from $L^\infty(\mathscr{A})$, since this is easily seen to be true in $C(Y)$. Hence by 7.1 and 5.6,

PROPOSITION 7.2. *Under the inclusion order on $\mathscr{C}\ell(Y)$ and the order inherited by $I^\infty(\mathscr{A})$ from $L^\infty(\mathscr{A})$, $I^\infty(\mathscr{A})$ is completely isomorphic to $\mathscr{C}\ell(Y)$.*

The last statements perhaps requires the clarification that *within* $\mathscr{C}\ell(Y)$ the inclusion order defines the lattice operations.

7.3. $\mathscr{M}/\chi^{-1}(0)$ *and* $I^\infty(\mathscr{A})$

Let $[E]$ denote the equivalence class in $\mathscr{M}/\chi^{-1}(0)$ defined by $E \in \mathscr{M}$ and define $\chi_0\colon \mathscr{M}/\chi^{-1}(0) \to I^\infty(\mathscr{A})$ by $\chi_0[E] = \chi(E)$, using 3.6. Under the operations $\cap$ and $\triangle$ induced on $\mathscr{M}/\chi^{-1}(0)$ from $\mathscr{M}$, and $\oplus$ defined in 7.2 for $I^\infty(\mathscr{A})$, χ_0 is a homomorphism by 3.6 and clearly $1 - 1$. To see that χ_0 is onto, if $\phi \in I^\infty(\mathscr{A})$, let $E = \{\hat{\phi} \neq 0\} \subset Y \subset S$. By 5.16 $E \in \mathscr{M}$ and by 5.6 $\chi_0(E) = \chi(E) = \phi$. Hence by 7.1

PROPOSITION 7.3. *The quotient $\mathscr{M}/\chi^{-1}(0)$ of the σ-algebra $\mathscr{M}$ is completely isomorphic to $I^\infty(\mathscr{A})$ and $\mathscr{C}\ell(Y)$ under the operations of* 7.2.

Turning now to the extreme points $\mathscr{E}$ of the positive unit ball B^+ in $L^\infty(\mathscr{A})$, the linear isomorphism $\phi \to \hat{\phi}$ on $L^\infty(\mathscr{A})$ onto $C(Y)$ takes B^+ onto the positive unit ball $B(Y)$ in $C(Y)$ (5.6) and it is well known that $\phi \leftrightarrow \hat{\phi}$ preserves extreme points. It is known too, or easily seen, that the extreme points of $B(Y)$ are exactly the functions χ_E, $E \in \mathscr{C}\ell(Y)$. This establishes thru 7.3 the identity of $\mathscr{E}$ and $I^\infty(\mathscr{A})$ as subsets of $L^\infty(\mathscr{A})$ and so using 7.2.

PROPOSITION 7.4. *Under Arens multiplication and $\oplus$ in $L^\infty(\mathscr{A})$, $\mathscr{E}$ is a complete Boolean algebra completely isomorphic to $I^\infty(\mathscr{A})$, $\mathscr{M}/\chi^{-1}(0)$ and $\mathscr{C}\ell(Y)$.*

Notice that B^+ is weak * closed, hence compact, in $L^\infty(\mathscr{A})$ and then that, modifying the argument in 6.2 slightly, if $\phi \in B^+$, then $\phi = \inf\{\sum_{k=1}^{n} (k/n)\, \chi(E_k)$: $E_k = \{k/n < \hat{\phi} \leqslant (k+1)/n\}$ in $L^\infty(\mathscr{A})$, and ϕ is also the β-limit of the directed family $\{s: s = \int_Y \hat{s}\, d\chi, \hat{s} = \sum_{i=1}^{n} a_i \chi_{E_i}, \sum_{i=1}^{n} a_i = 1, \hat{s} \geqslant \hat{\phi}\}$, both stronger conclusions than the Krein–Milman theorem allows.

Now we consider Cunningham's class $\mathscr{P} = \mathscr{P}(\mathscr{A})$ of L-projections of $L^1(\mathscr{A})$. It is shown in [8] that the L-projections of any L-space commute and form a complete Boolean algebra under $PQ = P \circ Q$ and $P \oplus Q = P(I - Q) + Q(I - P)$ (symmetric difference). We shall assume only [8, 2.2] that L-projections commute to accomplish the desired identification. Define $\rho: \mathscr{C}\ell(Y) \to \mathscr{P}(\mathscr{A})$ by $\rho(E) = P_E$ where P_E is defined in 3.4 by $P_E \hat{\mu} = \hat{\mu}_{\chi(E)}$. The definition of measurability is exactly that $\hat{\mu} = P_E \hat{\mu} + P_{S \setminus E} \hat{\mu}$, hence $I - P_E = P_{S \setminus E}$, and for $\hat{\mu} \geqslant 0$, $\| \hat{\mu} \|_1 = \| P_E \hat{\mu} \|_1 + \|(I - P_E) \hat{\mu} \|_1$. But notice that $\| P_E \hat{\mu} \|_1 = \| P_E |\mu| \|_1$, since $\chi(E) \geqslant 0$ in $L^\infty(\mathscr{A})$, for any $\hat{\mu}$, which implies that $P_E \in \mathscr{P}$. Further, from 3.4 and 3.6 and the Arens multiplication in $L^\infty(\mathscr{A})$, it follows that ρ is a homomorphism of $(\mathscr{C}\ell Y, \cap, \triangle)$ into $(\mathscr{P}, \circ, \oplus)$.

THEOREM 7.4. *$\mathscr{C}\ell(Y)$ is completely isomorphic to $\mathscr{P}(\mathscr{A})$.*

Proof. By 7.1 we have only to show that ρ is onto and $1 - 1$. Now $P_E = P_F$ implies $\chi(E)\hat{\mu} = \chi(F)\hat{\mu}$ for all $\hat{\mu} \in L^1(\mathscr{A})$ which implies $E \triangle F \in \chi^{-1}(0)$ and both being clopen, implies $E = F$ by 7.3.

Now, given $P \in \mathscr{P}(\mathscr{A})$ we will find $E \in \mathscr{C}\ell(Y)$ such that $P = P_E$. If $a \in \mathscr{A}$, then $P_a \in \mathscr{P}(\mathscr{A})$ as noted just above and by [8, 2.2] all L-projections commute, whence $PP_a = P_a P$. Since P is a bounded operator in $L^1(\mathscr{A})$, 2.4 allows us to write $P = T_\phi$ for some $\phi \in L^\infty(\mathscr{A})$, whence $\phi^2 = \phi$ in $L^\infty(\mathscr{A})$. But then $\hat{\phi}$ has support $E \in \mathscr{C}\ell(Y)$. By 5.6 $\phi = \chi(E)$, whence $P\hat{\mu} = T_\phi \hat{\mu} = \hat{\mu}_\phi = \hat{\mu}_{\chi(E)} = P_E \hat{\mu}$ for all $\hat{\mu} \in L^1(\mathscr{A})$ completing the proof.

Remark 7.4. One can show alternatively that $E = \overline{\bigcap_{U \in Q} U^0} \cap Y$ where $Q = \{U \subset S: U \text{ is open in } S \text{ and } P_U P = P\}$ and the closure/interior is taken in the topology on Y.

Remark 7.5. It follows from 2.2, that the Boolean suprema in $I^\infty(\mathscr{A})$ or $\mathscr{P}(\mathscr{A})$ coincide respectively with the limit in the β topology on $L^\infty(\mathscr{A})$ or s.o.t. topology in $\mathscr{B}$ (2.4).

A slightly weaker and alternative, but pertinent remark, is that the isomorphisms of $\mathscr{C}\ell(Y)$, $\mathscr{M}/\chi^{-1}(0)$ or $\mathscr{P}(\mathscr{A})$ into $\mathscr{E} = I^\infty(\mathscr{A}) \subset L^\infty(\mathscr{A})$ all define vector measures from these Boolean algebras into $L^\infty(\mathscr{A})_\beta$ because of 5.20.

We conclude with a discussion of a pernicious anamoly in measure theory; see also Cunningham's remark at the close of [8, Section 5]. One way to state it is as follows. The complete Boolean algebra $\mathscr{C}\ell(Y)$ has Stone space βY, whence $\mathscr{S}(\mathscr{C}\ell(Y))$ can be regarded as a subset of $C(Y) \simeq C(\beta Y)$ and we have a topology β defined on $\mathscr{S}(\mathscr{C}\ell(Y))$, or on $C(Y)$, as usual. We also have the topology $\bar{\beta}$ defined on $C(Y)$ in 5.20. Since $C(Y)_{\bar{\beta}}$ is topologically isomorphic to $L^\infty(\mathscr{A})$, $C(Y)'_{\bar{\beta}} = L^1(\mathscr{A}) \simeq \eta(\mathscr{C}\ell(Y), \mathscr{R})$ (5.22) and $C(Y)'_\beta = L^1(\mathscr{C}\ell(Y)) \supset \eta(\mathscr{C}\ell(Y), \mathscr{R})$. This implies the following inequalities: (i) $\bar{\beta} \leqslant \beta$, (ii) $L^\infty(\mathscr{A}) \subset L^\infty(\mathscr{C}\ell(Y))$,

(iii) $L^1(\mathscr{A}) \subset L^1(\mathscr{C}\ell(Y))$, and the anamoly is that without further condition, equality in (i), (ii), or (iii) does not occur. A second form of this anamoly is that if $\mathscr{A}$ begins as a complete Boolean algebra, then: (iv) $\mathscr{A} \subset \mathscr{C}\ell(Y)$, with equality again requiring further condition.

If η is a normal measure on a complete Boolean algebra $\mathscr{C}$ it is easy to prove that the support S_η of η coincides with the clopen image of some $c \in \mathscr{C}$; i.e., $S_\eta = c$ for some c. The dimension of $\mathscr{C}$ is defined to be the cardinal of any maximal p.w.d. subset B of $\mathscr{C}$ each of whose elements is the support of a normal measure; see [8, Lemma 6.4] for a proof that dim $\mathscr{C}$ is well-defined, which is principally the observation that $c\mathscr{C}$ is a c.c.c. algebra, where $c = S_\eta$, whence $B \cap c\mathscr{C}$ is at most countable. It is shown in [26, 6.7] that $L^1(\mathscr{C}) = \eta(\mathscr{C}, \mathscr{R})$ iff dim $\mathscr{C}$ is a cardinal of measure zero, at least when $\eta(\mathscr{C}, \mathscr{R})$ separates points of $\mathscr{S}(\mathscr{C})$ (i.e., β is Hausdorff). Hence

PROPOSITION 7.6. *Equality holds in* (i), (ii) *or* (iii) *iff* $\dim(\mathscr{C}\ell(Y))$ *is a cardinal of measure zero.*

Finally suppose that $\mathscr{A}$ is a complete Boolean algebra

PROPOSITION 7.7. $\mathscr{A} \simeq \mathscr{C}\ell(Y)$ *iff* $L^1(\mathscr{A}) = \eta(\mathscr{A}, \mathscr{R})$.

Proof. Since $L^1(\mathscr{A}) = \eta(\mathscr{C}\ell(Y), \mathscr{R})$ by 5.22, the necessity is clear. Conversely, suppose $L^1(\mathscr{A}) = \eta(\mathscr{A}, \mathscr{R})$. Then each $S_\mu = a$ for some $a \in \mathscr{A}$; of course, $a = e - \bigvee\{b: \mu(b) = 0\}$. Hence $\sigma = \hat{\sigma}$ on Y and Y is an open dense subset of S. In fact, if $a \in \mathscr{A}$, then $a = \overline{a \cap Y}^\sigma$. Hence the map $\pi: \mathscr{A} \to \mathscr{C}\ell(Y)$ by $\pi(a) = a \cap Y$ is $1 - 1$. It is onto because if $C \in \mathscr{C}\ell(Y)$, then $C = \cup S_\mu = \bigcup_{a \in B}\{a: a = S_\mu\}$ for some $B \subset \mathscr{A}$. Now $a = \bigvee\{b: b \in B\} \in \mathscr{A}$ and $a \cap Y = C$. Finally, $\pi(a \wedge b) = \pi(a) \wedge \pi(b)$ and $\pi(a \vee b) = (a \cup b) \cap Y = \pi(a) \vee \pi(b)$ since $\pi(a)$ is clopen. By 7.1 $\mathscr{A} \simeq \mathscr{C}\ell(Y)$.

Hence dim$\mathscr{A}$ is a cardinal of measure zero iff dim $\mathscr{C}\ell(Y)$ is, since, [26, 6.7] and 7.6, these are equivalent to the respective L^1 spaces being the respective spaces of normal measures.

ACKNOWLEDGMENTS

One of the authors (Dennis Sentilles) was supported in this work by NSF MCS76-06724, and wishes to also express his thanks to the Mathematics Department of the University of North Carolina at Chapel Hill for its hospitality and support during his visit in 1976–77.

REFERENCES

1. R. BARTLE, A general bilinear vector integral, *Studia Math.* **15** (1956), 337–352.
2. R. C. BUCK, Bounded continuous functions on a locally compact space, *Michigan Math. J.* **5** (1958), 95–104.

3. I. Chitescu, The conjugate space of the space of measures, *Rev. Roumaine Math. Pures Appl.* **21** (1976), 1313–1316.
4. H. S. Collins, Strict weighted, and mixed topologies and applications, *Advances in Math.* **19** (1976), 207–237.
5. H. S. Collins, On the space $l^\infty(S)$ with the strict topology, *Math. Z.* **106** (1968), 361–373.
6. J. B. Conway, The strict topology and compactness in the space of measures II, *Trans. Amer. Math. Soc.* **126** (1967), 474–486.
7. J. B. Cooper, Mixed topologies and applications to functional analysis, preprint.
8. F. Cunningham, Jr., L-structure in L-spaces, *Trans. Amer. Math. Soc.* (1960), 274–299.
9. J. Dixmier, Sur certaines espaces considérès par M. H. Stone, *Summa. Bracil. Math.* **2** (1951); fasc. 11, 151–182.
10. N. Dunford and J. Schwartz, "Linear Operators, Part I," Interscience, New York, 1958.
11. D. H. Fremlin, "Topological Riesz Spaces and Measure Theory," Cambridge Univ. Press, London, 1974.
12. H. Gordon, The maximal ideal space of a ring of measurable functions, *Amer. J. Math.* **88** (1966), 827–843.
13. W. H. Graves and S. M. Molnar, A universal measure, *Studies in Appl. Math.* **54** (1975), 117–134.
14. W. H. Graves, On the theory of vector measures, *Mem. Amer. Math. Soc.* **195** (1977).
15. P. R. Halmos, "Lectures on Boolean Algebras," Van Nostrand, Princeton, N. J., 1963.
16. P. R. Halmos, "Measure Theory," Van Nostrand, Princeton, N. J., 1950.
17. J. Henry and D. C. Taylor, The $\bar{\beta}$ topology for W^*-algebras, *Pacific J. Math.* **60** (1975), 123–139.
18. S. Kaplan, The second dual of the space of continuous functions and the Riemann integral, *Illinois J. Math.* **12** (1968), 283–302.
19. F. E. J. Linton, Functional measure theory, *in* "Proc. Conf. on Functional Analysis at U. Cal., Irvine," (B. Gelbaum, Ed.), pp. 36–49, Thompson Book Co., Washington, D.C., 1967.
20. J. C. Oxtoby, "Measure and Category," Springer–Verlag, New York, 1971.
21. G.-C. Rota, On the combinatorics of the Euler characteristic, *in* "Studies in Pure Math.," (L. Mirsky, Ed.), pp. 221–233, Academic Press, London, 1971.
22. A. P. Robertson and W. P. Robertson, "Topological Vector Spaces," Cambridge Univ. Press, New York, 1964.
23. S. Scheinberg, Topologies which generate a complete measure algebra, *Advances in Math.* **7** (1971), 231–239.
24. D. Sentilles, Bounded continuous functions on completely regular spaces, *Trans. Amer. Math. Soc.* **168** (1972), 311–336.
25. D. Sentilles, An L^1-space for Boolean algebras and semi-reflexivity of spaces $L^\infty(X, \Sigma, \mu)$, *Bull. Amer. Math. Soc.* **81** (1975), 1096–1098.
26. D. Sentilles, An L^1-space for Boolean algebras and semi-reflexivity of spaces $L^\infty(X, \Sigma, \mu)$, *Trans. Amer. Math. Soc.* **226** (1977), 1–37.
27. D. Sentilles and D. C. Taylor, Factorizations in Banach algebras and the general strict topology, *Trans. Amer. Math. Soc.* **142** (1969), 141–152.
28. R. Sikorski, "Boolean algebras," Springer–Verlag, Berlin, 1960.
29. W. H. Graves and D. Sentilles, Decompositions and extensions of vector measures: A universal approach, *in* "Proceedings of Conference on Vector Space Measures and Their Applications, Trinity College, Dublin, 1977," Springer–Verlag, New York, in press.